Eugene E Lundquist

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BASIC DIFFERENTIAL EQUATIONS IN GENERAL THEORY OF ELASTIC SHELLS

By V. S. Vlasov

Translation

"Osnovnye Differentsialnye Uravnenia Obshche Teorii Uprugikh Obolochek"
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BASIC DIFFERENTIAL EQUATIONS IN GENERAL THEORY OF ELASTIC SHELLS*

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1. Coordinates. - The shell shall be considered as a three-dimensional continuous medium; for the coordinate surface, the middle surface of the shell shall be assumed parallel to the bounding surfaces. Let α and β be the curvilinear orthogonal coordinates of this surface, coinciding with the lines of principal curvatures, and γ the distance along the normal from the point (α,β) of the coordinate surface to any point (α,β,γ) of the shell (fig. 1).

The square of the line element in spatial orthogonal curvilinear coordinates is given by the formula

$$ds^2 = \frac{d\alpha^2}{h_1{}^2} + \frac{d\beta^2}{h_2{}^2} + \frac{d\gamma^2}{h_3{}^2} \tag{1.1}$$

where $h_1 = h_1(\alpha,\beta,\gamma)$, $h_2 = h_2(\alpha,\beta,\gamma)$, and $h_3 = h_3(\alpha,\beta,\gamma)$ are the so-called differential parameters of the first kind representing for the chosen coordinates given functions of α,β,γ.

In the triorthogonal system of coordinates chosen as indicated:

$$\left.\begin{aligned} h_1 &= \frac{1}{A(1+k_1\gamma)} \\ h_2 &= \frac{1}{B(1+k_2\gamma)} \\ h_3 &= 1 \end{aligned}\right\} \tag{1.2}$$

where $A = A(\alpha,\beta)$ and $B = B(\alpha,\beta)$ are the coefficients of the first quadratic form; and where $k_1 = k_1(\alpha,\beta)$ and $k_2 = k_2(\alpha,\beta)$ are the principal curvatures of the coordinate surface on the lines corresponding to β = constant and α = constant, respectively.

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*"Osnovnye Differentsialnye Uravnenia Obshche Teorii Uprugikh Obolochek." Prikladnaia Matematika I Mekhanika. Vol. 8, 1944, pp. 109-140.

The magnitudes h_1, h_2, A, B, k_1, and k_2 are connected by the relations

$$\left.\begin{aligned}\frac{\partial}{\partial\alpha}\left(h_1\frac{\partial h_2^{-1}}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(h_2\frac{\partial h_1^{-1}}{\partial\beta}\right) &= -k_1k_2AB\\ h_1\frac{\partial h_2^{-1}}{\partial\alpha} &= \frac{1}{A}\frac{\partial B}{\partial\alpha}\\ h_2\frac{\partial h_1^{-1}}{\partial\beta} &= \frac{1}{B}\frac{\partial A}{\partial\beta}\end{aligned}\right\} \tag{1.3}$$

obtained from the equations of Lamb for the differential parameters h_1, h_2, and h_3 defined by equations (1.2) and from the equations of Codazzi

$$\left.\begin{aligned}\frac{\partial}{\partial\alpha}(k_2B) &= k_1\frac{\partial B}{\partial\alpha}\\ \frac{\partial}{\partial\beta}(k_1A) &= k_2\frac{\partial A}{\partial\beta}\end{aligned}\right\} \tag{1.4}$$

From equations (1.2), (1.3), and (1.4) follow the equalities:

$$\left.\begin{aligned}h_1\frac{\partial h_2^{-1}}{\partial\alpha}\frac{\partial h_1^{-1}}{\partial\gamma} &= \frac{\partial^2 h_2^{-1}}{\partial\alpha\partial\gamma}\\ h_2\frac{\partial h_1^{-1}}{\partial\beta}\frac{\partial h_2^{-1}}{\partial\gamma} &= \frac{\partial^2 h_1^{-1}}{\partial\beta\partial\gamma}\end{aligned}\right\} \tag{1.5}$$

2. Fundamental Equations of Three-Dimensional Problem of Theory of Elasticity. - The six components of the strain tensor of a dense medium are determined, in the system of coordinates assumed, by the equations

$$
\left.
\begin{aligned}
e_{\alpha\alpha} &= h_1 \frac{\partial u_\alpha}{\partial \alpha} + h_1 h_2 \frac{\partial h_1^{-1}}{\partial \beta} u_\beta + h_1 \frac{\partial h_1^{-1}}{\partial \gamma} u_\gamma \\
e_{\beta\beta} &= h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 \frac{\partial h_2^{-1}}{\partial \alpha} u_\alpha + h_2 \frac{\partial h_2^{-1}}{\partial \gamma} u_\gamma \\
e_{\gamma\gamma} &= \frac{\partial u_\gamma}{\partial \gamma} \\
e_{\alpha\beta} &= \frac{h_1}{h_2} \frac{\partial}{\partial \alpha} (h_2 u_\beta) + \frac{h_2}{h_1} \frac{\partial}{\partial \beta} (h_1 u_\alpha) \\
e_{\beta\gamma} &= \frac{1}{h_2} \frac{\partial}{\partial \gamma} (h_2 u_\beta) + h_2 \frac{\partial u_\gamma}{\partial \beta} \\
e_{\gamma\alpha} &= \frac{1}{h_1} \frac{\partial}{\partial \gamma} (h_1 u_\alpha) + h_1 \frac{\partial u_\gamma}{\partial \alpha}
\end{aligned}
\right\} \qquad (2.1)
$$

where $u_\alpha = u_\alpha(\alpha,\beta,\gamma)$, $u_\beta = u_\beta(\alpha,\beta,\gamma)$, and $u_\gamma = u_\gamma(\alpha,\beta,\gamma)$ are the components of the displacement vector of the point (α,β,γ) on the axes of the orthogonal trihedron, the vertex of which is at the point (α,β,γ) and the faces of which coincide with the planes tangent to the surfaces α = constant, β = constant, and γ = constant. The positive direction for the displacements corresponds to the direction of increase of the coordinates α, β, γ.

Equations (2.1) are obtained from the general formulas of the theory of elasticity given for example in the book of L. S. Leibenson (reference 1) for $h_3 = 1$.

The equations of equilibrium of the general problem of the theory of elasticity in the coordinates of the shell are presented in the form

$$\frac{\partial}{\partial\alpha}\left(\frac{\sigma_\alpha}{h_2}\right) - \frac{\partial h_2^{-1}}{\partial\alpha}\sigma_\beta + h_1\frac{\partial}{\partial\beta}\left(\frac{\tau_{\alpha\beta}}{h_1^2}\right) + h_1\frac{\partial}{\partial\gamma}\left(\frac{\tau_{\alpha\gamma}}{h_1^2 h_2}\right) + \frac{p_\alpha}{h_1 h_2} = 0$$

$$\frac{\partial}{\partial\beta}\left(\frac{\sigma_\beta}{h_1}\right) - \frac{\partial h_1^{-1}}{\partial\beta}\sigma_\alpha + h_2\frac{\partial}{\partial\alpha}\left(\frac{\tau_{\beta\alpha}}{h_2^2}\right) + h_2\frac{\partial}{\partial\gamma}\left(\frac{\tau p_\gamma}{h_2^2 h_1}\right) + \frac{p_\beta}{h_1 h_2} = 0$$

$$-\frac{1}{h_2}\frac{\partial h_1^{-1}}{\partial\gamma}\sigma_\alpha - \frac{1}{h_1}\frac{\partial h_2^{-1}}{\partial\gamma}\sigma_\beta + \frac{\partial}{\partial\alpha}\left(\frac{\tau_{\gamma\alpha}}{h_2}\right) + \frac{\partial}{\partial\beta}\left(\frac{\tau_{\gamma\beta}}{h_1}\right) + \frac{\partial}{\partial\gamma}\left(\frac{\sigma_\gamma}{h_1 h_2}\right) + \frac{p_\gamma}{h_1 h_2} = 0$$

(2.2)

where σ and τ are the normal and tangential stresses. The subscripts denote, for normal stresses, the direction of the outward (in the direction of increase of the corresponding coordinates) normal to the corresponding surface; for the tangents, they denote the surface of action of these stresses taken in pairs from the conditions of their reciprocity. The components of the stress tensor are considered positive if, when applied to the surface with positive outer normal, they are directed toward increasing coordinates. The magnitudes p_α, p_β, and p_γ in equations (2.2) are the components of the vector of intensity of the volume forces. Equations (2.2) are obtained from the general equations of the theory of elasticity given for example by Love (reference 2) for $h_3 = 1$.

In the theory of shells, the stresses σ_α, σ_β, $\tau_{\alpha\beta} = \tau_{\beta\alpha}$, applied normal to the section and lying in a plane tangent to v = constant, are determined from the six stress components expressed in terms of the strains. The remaining three components of the stresses are found from the conditions of equilibrium. From Hooke's law, only the three relations referring to the stresses σ_α, σ_β, and $\tau_{\alpha\beta}$ are retained; these relations are given in the form

$$\left.\begin{aligned}\sigma_\alpha &= (\lambda+2\mu)\ \Delta - 2\mu\ (e_{\beta\beta}+e_{\gamma\gamma})\\ \sigma_\beta &= (\lambda+2\mu)\ \Delta - 2\mu\ (e_{\alpha\alpha}+e_{\gamma\gamma})\\ \tau_{\alpha\beta} &= \tau_{\beta\alpha} = \mu e_{\alpha\beta}\end{aligned}\right\} \qquad (2.3)$$

where Δ is the volumetric dilation and λ and μ are the coefficients of elasticity of Lamé.

The equations of equilibrium (2.2), on the basis of equations (2.3) and (2.1) after a number of transformations using, where required, the relations (1.3), (1.4) and (1.5), are given in the form

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{1}{h_2}\frac{\partial\Delta}{\partial\alpha}-2\mu\frac{1}{h_1}\frac{\partial X}{\partial\beta}+2\mu ABKu_\alpha-2\mu\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\frac{\partial u_\gamma}{\partial\alpha}\right)+h_1\frac{\partial}{\partial\gamma}\left(\frac{\tau_{\alpha\gamma}}{h_1{}^2h_2}\right)+\frac{p_\alpha}{h_1h_2}=0\\
&(\lambda+2\mu)\frac{1}{h_1}\frac{\partial\Delta}{\partial\beta}+2\mu\frac{1}{h_2}\frac{\partial X}{\partial\alpha}+2\mu ABKu_\beta-2\mu\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\frac{\partial u_\gamma}{\partial\beta}\right)+h_2\frac{\partial}{\partial\gamma}\left(\frac{\tau_{\beta\gamma}}{h_2{}^2h_1}\right)+\frac{p_\beta}{h_1h_2}=0\\
&-2(\lambda+2\mu)(H+K\gamma)AB\Delta+2\mu\left[\frac{\partial}{\partial\alpha}(Bk_2u_\alpha)+\frac{\partial}{\partial\beta}(Ak_1u_\beta)+2ABKu_\gamma\right]+\\
&4\mu AB(H+K\gamma)\frac{\partial u_\gamma}{\partial\gamma}+\frac{\partial}{\partial\alpha}\left(\frac{\tau_{\gamma\alpha}}{h_2}\right)+\frac{\partial}{\partial\beta}\left(\frac{\tau_{\gamma\beta}}{h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\sigma_\gamma}{h_1h_2}\right)+\frac{p_\gamma}{h_1h_2}=0
\end{aligned}\right\}\qquad(2.4)$$

where $K = K(\alpha,\beta)$ and $H = H(\alpha,\beta)$ are the Gaussian and mean curvatures of the coordinate surface, respectively.

$$\left.\begin{aligned}
K &= k_1k_2\\
H &= \frac{1}{2}(k_1+k_2)
\end{aligned}\right\}\qquad(2.5)$$

The volumetric expansion and the normal component (the projection on the normal to the surface γ = constant) of the elementary rotation of the shell are denoted by $\Delta = \Delta(\alpha,\beta,\gamma)$ and $2X = 2X(\alpha,\beta,\gamma)$, respectively. In the following discussion $2X$ shall be denoted simply the normal rotation. The volumetric expansion and normal rotation are determined in terms of the displacements u_α, u_β, and u_γ in the coordinates of the shell by the formulas

$$\left.\begin{aligned} \Delta &= h_1h_2\left[\frac{\partial}{\partial\alpha}\left(\frac{u_\alpha}{h_2}\right)+\frac{\partial}{\partial\beta}\left(\frac{u_\beta}{h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{u_\gamma}{h_1h_2}\right)\right] \\ 2\chi &= h_1h_2\left[\frac{\partial}{\partial\alpha}\left(\frac{u_\beta}{h_2}\right)-\frac{\partial}{\partial\beta}\left(\frac{u_\alpha}{h_1}\right)\right] \end{aligned}\right\} \tag{2.6}$$

Equations (2.4) in orthogonal coordinates for $h_3 = 1$ are the general equations of equilibrium of an elastic body. The first two of these equations express the tangential equilibrium of the three-dimensional element $d\alpha d\beta d\gamma/h_1h_2$ of the shell, that is, the equilibrium of this element in the plane tangent to the surface γ = constant. The last equation refers to the equilibrium of this element in the direction of the outer normal to the surface γ = constant. Equations (2.4) differ from the equation of the general problem of the theory of elasticity in displacements or strains in the fact that each of them contains both static and kinematic magnitudes.

3. Displacements and Strains of the Shells. - The theory of shells is based, as is known, on the hypothesis of Kirchhoff-Love according to which a rectilinear element normal to the middle surface of the shell remains, after deformation, rectilinear normal to this surface and of the same length. This hypothesis is equivalent to the assumption

$$e_{\alpha\gamma} = e_{\beta\gamma} = e_{\gamma\gamma} = 0 \tag{3.1}$$

and leads, for the displacements u_α, u_β, and u_γ of an arbitrary point (α,β,γ) to the formulas

$$\left.\begin{aligned} u_\alpha &= (1+k_1\gamma)\ u - \frac{\gamma}{A}\frac{\partial w}{\partial\alpha} \\ u_\beta &= (1+k_2\gamma)\ v - \frac{\gamma}{B}\frac{\partial w}{\partial\beta} \\ u_\gamma &= w \end{aligned}\right\} \tag{3.2}$$

where $u = u(\alpha,\beta)$ and $v = v(\alpha,\beta)$ are the tangential displacements (in the direction of the tangents to the lines β = constant and α = constant) of the point (α,β) of the coordinate surface, and

$w = w(\alpha,\beta)$ is the displacement of the same point in the direction normal to this surface, positive in the direction of increasing coordinate γ (fig. 2).

For the purpose of presenting a more accurate theory valid not only for shells of medium thickness but also for thick shells, another hypothesis, which is a generalization of that of Kirchhoff-Love, will be used.

It shall be considered that each of the three components u_α, u_β, and u_γ is represented as a function of γ by a linear law, setting

$$\left.\begin{aligned} u_\alpha &= (1+k_1\gamma)\ u - \frac{\gamma}{A}\frac{\partial w}{\partial \alpha} \\ u_\beta &= (1+k_2\gamma)\ v - \frac{\gamma}{B}\frac{\partial w}{\partial \beta} \\ u_\gamma &= w + \gamma w^* \end{aligned}\right\} \qquad (3.3)$$

where u, v, and w have the same values as in equations (3.2); $w^* = w^*(\alpha,\beta)$ is a magnitude that depends, like u, v, and w, only on the two variables α,β and is the relative elongation of a normal element of the shell (constant under the assumption made here over the entire length of this element). It is easy to see that with equations (3.3) equations (3.1), which express the fundamental hypothesis of the present theory of shells, do not apply. With the introduction of the deformation of elongation w^* of a normal element of the shell, all the six components of the strain tensor (2.1) receive values different from zero.

Equations (3.2) establish the kinematic model of the deformed state of the shell. This state, in the general case, is made up of two states of which the first is determined only on the tangential displacement u,v of the point of the coordinate surface (w, w^* in this case being equal to zero) and in the second only by the normal displacement w and the elongation w^* (u,v in this case being equal to zero). The deformation of the shell determined only by the tangential displacements u,v shall be denoted the tangential deformation for briefness. This deformation is characterized by the fact that an arbitrary point of the surface γ = constant after deformation does not go beyond the limits of this surface as a two-dimensional space. An elementary layer of the shell $d\gamma$ for a tangential deformation does not change its shape and position in space and undergoes deformations of length and shear at the surface γ = constant as a two-dimensional space (in the general case for $K \neq 0$ non-Euclidean). A deformation of the second kind determined only by the normal displacements w and the elongation w^* will be called a normal deformation of the shell. For this deformation, an arbitrary point (α,β,γ) of the surface γ = constant passes with respect to this surface into the third dimension. A normal deformation is accompanied by a change in shape of the surface.

In setting up the kinematic model determined by equations (3.3) for all six components of the deformation tensor, by virtue of equations (2.1) and (1.2), a definite law of variation with thickness of the shell is obtained.

Representation of $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ in the form of series in the variable γ gives

$$\left.\begin{aligned} e_{\alpha\alpha} &= \epsilon_1 + \sum \chi_{1n}\,\gamma^n \\ e_{\beta\beta} &= \epsilon_2 + \sum \chi_{2n}\,\gamma^n \\ e_{\alpha\beta} &= \omega + \sum \tau_n\,\gamma^n \\ (n &= 1,2,3, \ldots) \end{aligned}\right\} \tag{3.4}$$

where the coefficients of the series ϵ_1, ϵ_2, ω, χ_{1n}, χ_{2n}, and τ_n each depends only on the displacements u, v, and w^* of the point (α,β) of the coordinate surface. By substituting the displacements u_α, u_β, and u_γ determined by equations (3.3) on the right-hand sides of the corresponding equations (2.1) and then by representing the magnitudes h_1, h_2, and their ratios (direct and inverse) in the form of series

$$\left.\begin{aligned}
h_1 &= \frac{1}{A}\left(1 - k_1\gamma + k_1{}^2\gamma^2 - k_1{}^3\gamma^3 + \ldots\right)\\
h_2 &= \frac{1}{B}\left(1 - k_2\gamma + k_2{}^2\gamma^2 - k_2{}^3\gamma^3 + \ldots\right)\\
\frac{h_1}{h_2} &= \frac{B}{A}\left[1 - (k_1 - k_2)(\gamma - k_1\gamma^2 + k_1{}^2\gamma^3 - \ldots)\right]\\
\frac{h_2}{h_1} &= \frac{A}{B}\left[1 + (k_1 - k_2)(\gamma - k_2\gamma^2 + k_2{}^2\gamma^3 - \ldots)\right]
\end{aligned}\right\} \quad (3.5)$$

and referring to the last two of relations (1.3) and relations (1.4), after a number of transformations for the coefficients of the series (3.4) the following equations are obtained:

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + k_1 w\\
\epsilon_2 &= \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{1}{B}\frac{\partial v}{\partial \beta} + k_2 w\\
\omega &= \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{u}{B}\right)\\
\chi_{1n} &= (-1)^{n-1}k_1^{n-1}\left[\frac{\partial k_1}{\partial \alpha}\frac{u}{A} + \frac{\partial k_1}{\partial \beta}\frac{v}{B} - k_1{}^2 w - \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + k_1 w^*\right]\\
\chi_{2n} &= (-1)^{n-1}k_2^{n-1}\left[\frac{\partial k_2}{\partial \alpha}\frac{u}{A} + \frac{\partial k_2}{\partial \beta}\frac{v}{B} - k_2{}^2 w - \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + k_2 w^*\right]\\
\tau_n &= (-1)^{n-1}\left\{(k_1 - k_2)\left[k_2{}^{n-1}\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - k_1{}^{n-1}\frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right] - \right.\\
&\qquad \left.\frac{k_1{}^{n-1} + k_2{}^{n-1}}{AB}\left(\frac{\partial^2 w}{\partial \alpha\,\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha}\right)\right\}
\end{aligned}\right\}$$

(3.6)

The remaining strains $e_{\alpha\gamma}$, $e_{\beta\gamma}$, and $e_{\gamma\gamma}$ depend only on w^*. After expanding them in a series in powers of γ,

$$\left.\begin{aligned} e_{\alpha\gamma} &= (\gamma - k_1\gamma^2 + k_1{}^2\gamma^3 - \ldots) \frac{1}{A}\frac{\partial w^*}{\partial \alpha} \\ e_{\beta\gamma} &= (\gamma - k_2\gamma^2 + k_2{}^2\gamma^3 - \ldots) \frac{1}{B}\frac{\partial w^*}{\partial \beta} \\ e_{\gamma\gamma} &= w^* \end{aligned}\right\} \tag{3.7}$$

In the following discussion, formulas for the volume expansion Δ and the elementary rotation 2χ will be required. When these magnitudes are also represented in the form of series in powers of γ

$$\left.\begin{aligned} \Delta &= \Delta_0 + \sum \Delta_n\gamma^n \\ \chi &= \chi_0 + \sum \chi_n\gamma^n \end{aligned}\right\} \tag{3.8}$$

Then by making use of equations (2.6) and (3.5) after a number of transformations, using relations (1.3) and (1.4) for the coefficients of the series (3.8), the following formulas are obtained:

$$
\left.
\begin{aligned}
&\Delta_0 = \epsilon_1 + \epsilon_2 + w^* = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu) + \frac{\partial}{\partial\beta}(Av)\right] + (k_1+k_2)\,w + w^* \\
&\Delta_n = x_{1n} + x_{2n} = (-1)^{n-1}\left\{\left(k_1^{\,n-1}\frac{\partial k_1}{\partial\alpha} + k_2^{\,n-1}\frac{\partial k_2}{\partial\alpha}\right)\frac{u}{A} + \right. \\
&\left(k_1^{\,n-1}\frac{\partial k_1}{\partial\beta} + k_2^{\,n-1}\frac{\partial k_2}{\partial\beta}\right)\frac{v}{B} - k_1^{\,n-1}\left[\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha}\right] - \\
&\left. k_2^{\,n-1}\left[\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}\right]\ (k_1^{\,n+1} + k_2^{\,n+1})w + (k_1^{\,n} + k_2^{\,n})w^*\right\} \\
&\chi_0 = \frac{1}{2AB}\left[\frac{\partial}{\partial\alpha}(Bv) - \frac{\partial}{\partial\beta}(Au)\right] \\
&\chi_n = (-1)^n\left\{\frac{k_1-k_2}{2}\left[k_2^{\,n-1}\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + k_1^{\,n-1}\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)\right] + \right. \\
&\left. \frac{k_1^{\,n-1} - k_2^{\,n-1}}{2AB}\left(\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)\right\}
\end{aligned}
\right\} \quad (3.9)
$$

4. Analysis of Kinematic Relations. Corrections and Additions to Theory of Love. - Equations (3.4) and (3.6) for the components of the deformations have a common character and were obtained in correspondence with the hypothesis (3.3) assumed for the displacements. For $w^* = 0$ from equations (3.3, 3.4, and 3.6), there are obtained equations for the displacements and the deformations of the shell having an inextensible normal element and following the hypothesis of Love (3.2). The magnitudes χ_{11}, χ_{21}, and τ_1, defined by equations (3.6) for $n = 1$ and $w^* = 0$ and the first two representing bending deformations (variation of the principal curvatures k_1 and k_2) and the third the torsional deformation, differ from the corresponding magnitudes χ_1, χ_2, and τ , which were used by Love. By setting $w^* = 0$ in the last three equations of (3.6),

$$\left.\begin{aligned}
\chi_{11} &= \frac{\partial k_1}{\partial \alpha}\frac{u}{A} + \frac{\partial k_1}{\partial \beta}\frac{v}{B} - k_1^{\,2}w - \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} \\
\chi_{21} &= \frac{\partial k_2}{\partial \alpha}\frac{u}{A} + \frac{\partial k_2}{\partial \beta}\frac{v}{B} - k_2^{\,2}w - \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} \\
\tau_1 &= (k_1-k_2)\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right] - \frac{2}{AB}\left(\frac{\partial^2 w}{\partial \alpha\, \partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha}\right)
\end{aligned}\right\} \quad (4.1)$$

From these equations, it follows that for tangential deformations (in the case $w = 0$) the changes in curvature χ_{11}, and χ_{21} are determined as linear algebraic expressions relative to the displacements u and v with coefficients proportional to the partial derivatives of the principal curvatures k_1, and k_2 of the undeformed surface. The expression for the torsional deformation will, as is to be expected, be symmetrical with respect to the coordinates. The same properties, as seen from equations (3.6), are possessed also by the remaining components χ_{1n}, χ_{2n}, and τ_n for $n = 2, 3, 4, \ldots$. In particular, for the spherical shell a result is obtained that generalizes in a certain sense the theory of the bending deformation of a plate as based on the hypothesis of Kirchhoff. This result can be formulated in the following theorems:

Theorem I. - The deformations of elongation and shear $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ and the volume deformation Δ of a spherical shell in the case of tangential deformations (that is, for $w = 0$) are uniformly distributed over the thickness of the shell (do not depend on γ) and are determined only by the deformations of elongation and shear ϵ_1, ϵ_2, and ω of the middle surface. An exception to the uniform distribution of the magnitudes $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ over the thickness of the shell arises only as a result of normal displacements. A change in the shape of the spherical shell characterized by the parameters of the change in curvature χ_{11}, χ_{21}, and τ_1 is due only to the normal displacement w.

Theorem II. - The normal rotation 2χ of the spherical shell is determined only by the tangential deformation (the variables u and v) and remains constant over the thickness of the shell. In the case of normal deformation, the normal rotation 2χ is equal to zero.

This result, obtained on the basis of the analysis of the general formulas of the preceding section for the spherical shell, may be arrived at directly in the following manner. It is assumed that the spherical shell as a deformed body is, at one of its bounding surfaces (for example, the inner), in contact with a rigid spherical base so that an arbitrary point of the shell can be freely displaced along the surface of this base without going outside the limits of this surface. Such a model corresponds to the case of tangential deformation of the shell. Now at the point (α, β) some normal section of the shell, in general arbitrary with respect to the chosen coordinate lines α and β, is assumed. The linear normal element, as the shell passes into the deformed state, remains, by the Love hypothesis, normal to the base surface and takes on a new position determined by the rotation of this element with respect to the center of curvature (in the case of a sphere, common for all normal elements). Let $M_1'M_2'$ be the projection on the plane of the chosen section of the element M_1M_2 after deformation (fig. 3). Further let $\xi = M_1M_1'$ denote the projection on the plane of this section of the vector of the total displacement of the lower point M_1 of the element. Then the displacement $\xi_\gamma = MM'$ of an arbitrary point M of the element M_1M_2 in the plane of the chosen section will be equal to

$$\xi_\gamma = \xi(1 + k\gamma) \tag{4.2}$$

where $k = 1/R$ is the curvature of the inner surface of the shell and γ is the distance of this surface to the point M considered. The corresponding elongation of the tangential element $ds = (1 + k\gamma)\, Rd\varphi$ is determined by the equation

$$e = \frac{\partial \xi_\gamma}{\partial s} = \frac{1}{R(1 + k\gamma)} \frac{\partial}{\partial \varphi}\left[(1 + k\gamma)\xi\right] = \frac{\partial \xi}{R\partial \varphi} \tag{4.3}$$

From equations (4.2) and (4.3), it follows that whereas the tangential displacement ξ_γ of the spherical shell is a linear function of the coordinate γ, the deformation of elongation e does not depend on γ. The same result can also be obtained directly from equations (2.1) for the deformations $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ for the values entering the following formulas:

$$\left.\begin{aligned} k_1 &= k_2 = k = \text{constant} \\ Ah_1 &= Bh_2 = \frac{1}{1 + k\gamma} \\ u_\alpha &= (1 + k\gamma)u \\ u_\beta &= (1 + k\gamma)v \end{aligned}\right\} \tag{4.4}$$

In the same way on the basis of the second of equations (2.6), the second theorem can be proven.

In constructing this theorem, Love represents the components of the deformation $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ in the form of linear expressions relative to the parameter γ.

$$\left.\begin{aligned} e_{\alpha\alpha} &= \epsilon_1 + \chi_1\gamma \\ e_{\beta\beta} &= \epsilon_2 + \chi_2\gamma \\ e_{\alpha\beta} &= \omega + \tau\gamma \end{aligned}\right\} \tag{4.5}$$

and for the parameters of the change in curvature χ_1, χ_2, and τ gives the equations

$$\left.\begin{aligned} \chi_1 &= \frac{1}{A}\frac{\partial}{\partial\alpha}(k_1u) + \frac{1}{AB}\frac{\partial A}{\partial\beta}(k_2v) - \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} \\ \chi_2 &= \frac{1}{AB}\frac{\partial B}{\partial\alpha}(k_1u) + \frac{1}{B}\frac{\partial}{\partial\beta}(k_2v) - \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} \\ \tau &= \frac{1}{A}\frac{\partial}{\partial\alpha}(k_2v) - \frac{1}{A}\frac{\partial v}{\partial\alpha}k_1 - \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial A}{d\beta}\frac{\partial w}{\partial\alpha} \end{aligned}\right\} \tag{4.6}$$

These equations differ essentially from equations (4.1). The magnitudes χ_1, χ_2, and τ determined by Love as coefficients of the second members of equations (4.3) are in contradiction to the theorems just proven for the spherical shell. The difference noted here in the determination of the magnitudes χ_1, χ_2, and τ by equations (4.1) and (4.6) is explained by the fact that Love and other authors (in particular, Timoshenko (reference 3)), following Rayleigh, start from the assumption of the inextensibility of the middle surface. This assumption stands in certain contradiction with the geometry of extensible and flexible surfaces.

In recent years a number of papers have appeared that refine to a greater or less extent the theory of thin shells of Love. The most interesting and original of these are the investigations of Krauss (reference 4), N. A. Kilchevsky (reference 5), and A. I. Lurie (reference 6).

5. Fundamental Differential Equations of Equilibrium of Elastic Shells. - The general equations of a shell possessing deformable

normal elements is obtained by reducing the three-dimensional problem of the theory of elasticity to two-dimensional starting from equations (2.4), retaining in the series (3.8) the first three terms, and applying the principle of Lagrange corresponding to the kinematic hypothesis (3.3).

An element of the shell $AB\delta d\alpha d\beta$, having an infinitely small area $ABd\alpha d\beta$ on the middle surface and a finite length δ equal to the thickness of the shell, possesses according to the kinematic model seven degrees of freedom; namely, six degrees with respect to the displacements of the element (three linear and three angular) in space as a rigid body, and one degree characterized by the change in length of the element. Corresponding to these degrees of freedom seven equations of equilibrium must be obtained. Of these equations, the first six refer to the equilibrium of the element in space as a rigid body and the seventh may be obtained by equating to zero the work of all the external and internal forces of the element $AB\delta d\alpha d\beta$ against displacements and deformations corresponding to the unit elongation $w^* = 1$. It should be noted that the equations of equilibrium of an element may also be obtained on the basis of the principle of virtual displacements by equating to zero the sum of the work of all the forces (in the given case only the external, because the element is considered as a rigid body) for each of the six possible unit displacements.

By the method assumed here, one of the conditions of equilibrium of the element as a rigid body, namely the condition corresponding to the rotation of the element about the normal to the middle surface and given in the theory of Love, the sixth nondifferential (relative to the shearing forces and torsional moments) is satisfied identically because of the relation $\tau_{\alpha\beta} = \tau_{\beta\alpha}$ used in deriving the general equations (2.4).

Thus, starting from equations (2.4) and applying the principle of virtual displacements, it will be necessary to obtain for an element of the shell only six equations, one of which (called above the seventh) according to its physical meaning represents the generalized condition of equilibrium of the element $AB\delta d\alpha d\beta$ having a strain w^* expressed as a function of γ.

Substituting in the left sides of equations (2.4) the displacements u_α, u_β, and u_γ according to equations (3.3), the volume dilation and the normal increment Δ and X according to equations (3.8) (in which it is necessary to retain only the first three terms, that is, to γ^2 inclusive and reject the others) and

h_1 and h_2 according to equations (1.2), three expressions are obtained each of which contains terms with powers of γ up to the third inclusive.

The three magnitudes thus obtained represent, according to their physical meaning, the components along the axes of the movable trihedron on the surface γ = constant of the vector of the external force acting on the three-dimensional element of the shell $\frac{d\alpha d\beta}{h_1 h_2} d\gamma$ and expressed in terms of the kinematic magnitudes u, v, w, w*, Δ_0, Δ_1, Δ_2, X_0, X_1, and X_2 and static magnitudes $\tau_{\gamma\alpha}$, $\tau_{\gamma\beta}$, σ_γ, p_α, p_β, and p_γ. In passing to the two-dimensional element of the shell $AB\delta d\alpha d\beta$, the work of all the forces acting on this element and determined in this manner must be equal to zero on each of the five possible displacements as a rigid body.

Corresponding to these displacements and by virtue of hypothesis (3.3), each of the first two equations of (2.4) must be by $d\gamma$ and $\gamma d\gamma$ and the third by $d\gamma$, integrated with respect to γ between the limits $\gamma = -\frac{1}{2}\delta$ to $\gamma = +\frac{1}{2}\delta$, and the result equated in each case to zero. Thus five equations are obtained containing in addition to terms with the kinematic magnitudes u, v, w, w*, Δ_0, Δ_1, Δ_2, X_0, X_1, and X_2 also terms with the transverse forces N_1 and N_2 arising from the tangential stresses $\tau_{\gamma\alpha}$ and $\tau_{\gamma\beta}$.

$$\left.\begin{aligned} N_1 &= \frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\gamma\alpha}}{h_2}\, d\gamma \\ N_2 &= \frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\gamma\beta}}{h_1}\, d\gamma \end{aligned}\right\} \tag{5.1}$$

In order to obtain the sixth equation corresponding to the linear strain of w* of the normal element of the shell, the left side of the third equation in (2.4) must be multiplied by $\gamma d\gamma$ and the integral of this expression between the limits $\gamma = -\frac{1}{2}\delta$ to $\gamma = +\frac{1}{2}\delta$ equated to zero. When it is remembered that

$$\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\partial}{\partial\gamma}\left(\frac{\sigma_\gamma}{h_1h_2}\right)\gamma d\gamma = \left|\frac{\sigma_\gamma}{h_1h_2}\gamma\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} - \int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\sigma_\gamma}{h_1h_2} d\gamma \quad (5.2)$$

where the first term refers to the work of the external and the second to the work of the internal normal forces of the element $AB\delta$ for the normal displacements $u_\gamma = \gamma w^*$ for $w^* = 1$, determining $\tau_{\gamma\alpha}$, $\tau_{\gamma\beta}$, and σ_γ in terms of the deformations by the equations

$$\left.\begin{aligned} \tau_{\gamma\alpha} &= \mu(\gamma - k_1\gamma^2)\frac{1}{A}\frac{\partial w^*}{\partial\alpha} \\ \tau_{\gamma\beta} &= \mu(\gamma - k_2\gamma^2)\frac{1}{B}\frac{\partial w^*}{\partial\beta} \\ \sigma_\gamma &= \lambda(\Delta_0 + \Delta_1\gamma + \Delta_2\gamma^2) + 2\mu w^* \end{aligned}\right\} \quad (5.3)$$

and representing the remaining terms of the third equation of (2.4) in the form of a finite series in powers of γ, an equation is obtained in which the unknown will be only one of the kinematic magnitudes.

Thus there are six equations with respect to 12 functions, the four basic functions u, v, w, and w^*, the six functions Δ_0, Δ_1, Δ_2, X_0, X_1, and X_2 giving the coefficients of the first three terms of the series (3.8) and the two transverse forces N_1 and N_2.

These equations, upon eliminating the forces N_1 and N_2, reduce to four equations. Neglecting the small terms with $k^2{}_1\delta^2/12$, $k^2{}_2\delta^2/12$, and $k_1k_2\delta^2/12$ in the expressions $1 + k^2{}_1\delta^2/12$, $1 + k^2{}_2\delta^2/12$, and $1 + k_1k_2\delta^2/12$ finally

$$
\begin{aligned}
&(\lambda+2\mu)\,\frac{1}{A}\left(\frac{\partial\Delta_0}{\partial\alpha}+\frac{\delta^2}{6}\,H\,\frac{\partial\Delta_1}{\partial\alpha}+\frac{\delta^2}{12}\,\frac{\partial\Delta_2}{\partial\alpha}\right)-2\mu\,\frac{1}{B}\left(\frac{\partial\chi_0}{\partial\beta}+\frac{\delta^2}{12}\,k_1\,\frac{\partial\chi_1}{\partial\beta}+\frac{\delta^2}{12}\,\frac{\partial\chi_2}{\partial\beta}\right)+\\
&\qquad 2\mu k_2\left(k_1 u-\frac{1}{A}\,\frac{\partial w}{\partial\alpha}\right)-2\mu\,\frac{1}{A}\,\frac{\partial w^*}{\partial\alpha}+\frac{1}{\delta}\,(X-k_1 m_\beta)=0\\
&(\lambda+2\mu)\,\frac{1}{B}\left(\frac{\partial\Delta_0}{\partial\beta}+\frac{\delta^2}{6}\,H\,\frac{\partial\Delta_1}{\partial\beta}+\frac{\delta^2}{12}\,\frac{\partial\Delta_2}{\partial\beta}\right)+2\mu\,\frac{1}{A}\left(\frac{\partial\chi_0}{\partial\alpha}+\frac{\delta^2}{12}\,k_2\,\frac{\partial\chi_1}{\partial\alpha}+\frac{\delta^2}{12}\,\frac{\partial\chi_2}{\partial\alpha}\right)+\\
&\qquad 2\mu k_1\left(k_2 v-\frac{1}{B}\,\frac{\partial w}{\partial\beta}\right)-2\mu\,\frac{1}{B}\,\frac{\partial w^*}{\partial\beta}+\frac{1}{\delta}\,(Y+k_2 m_\alpha)=0\\
&(\lambda+2\mu)\,\frac{\delta^2}{12}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,k_2\,\frac{\partial\Delta_0}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\,k_1\,\frac{\partial\Delta_0}{\partial\beta}\right)\right]-2(\lambda+2\mu)\,H\Delta_0+\\
&(\lambda+2\mu)\,\frac{\delta^2}{12}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,\frac{\partial\Delta_1}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\,\frac{\partial\Delta_1}{\partial\beta}\right)\right]-(\lambda+2\mu)\,\frac{\delta^2}{6}\,(K\Delta_1+H\Delta_2)-\\
&\mu\,\frac{\delta^2}{6}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(k_1\,\frac{\partial\chi_0}{\partial\beta}\right)-\frac{\partial}{\partial\beta}\left(k_2\,\frac{\partial\chi_0}{\partial\alpha}\right)\right]+2\mu\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\,(Bk_2 u)+\frac{\partial}{\partial\beta}\,(Ak_1 v)\right]+\\
&\qquad\mu\,\frac{\delta^2}{6}\,\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\left[BK\left(k_1 u-\frac{1}{A}\,\frac{\partial w}{\partial\alpha}\right)\right]+\frac{\partial}{\partial\beta}\left[AK\left(k_2 v-\frac{1}{B}\,\frac{\partial w}{\partial\beta}\right)\right]\right\}+\\
&\qquad 4\mu\,(Kw+Hw^*)-\mu\,\frac{\delta^2}{3}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,k_2\,\frac{\partial w^*}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\,k_1\,\frac{\partial w^*}{\partial\beta}\right)\right]+\\
&\qquad\qquad\frac{1}{\delta}\left\{Z-\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\,(Bm_\beta)-\frac{\partial}{\partial\beta}\,(Am_\alpha)\right]\right\}=0\\
&-\lambda\Delta_0-(\lambda+\mu)H\,\frac{\delta^2}{3}\,\Delta_1-\lambda\,\frac{\delta^2}{12}\,\Delta_2+\mu\,\frac{\delta^2}{6}\,\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\left[Bk_2\left(k_1 u-\frac{1}{A}\,\frac{\partial w}{\partial\alpha}\right)+\right.\right.\\
&\left.\left.\frac{\partial}{\partial\beta}\left[Ak_1\left(k_2 v-\frac{1}{B}\,\frac{\partial w}{\partial\beta}\right)\right]\right\}+\mu\,\frac{\delta^2}{12}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,\frac{\partial w^*}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\,\frac{\partial w^*}{\partial\beta}\right)\right]-2\mu w^*+\frac{1}{\delta}\,Z^*=0
\end{aligned}
\tag{5.4}
$$

where $X = X(\alpha,\beta)$, $Y = Y(\alpha,\beta)$, and $Z = Z(\alpha,\beta)$ are the components on the axes of the movable trihedron of the vector of surface intensity of the load computed for the stresses $\tau_{\alpha\gamma}$, $\tau_{\beta\gamma}$, and σ_γ on the boundary surfaces $\gamma = \frac{1}{2}\,\delta$, $\gamma = -\frac{1}{2}\,\delta$ and for given volume forces p_α, p_β, and p_γ by the formulas

$$\left.\begin{aligned}
X &= \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{p_\alpha}{h_1h_2}\,d\gamma + \left|\frac{\tau_{\alpha\gamma}}{h_1h_2}\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\} \\
Y &= \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{p_\beta}{h_1h_2}\,d\gamma + \left|\frac{\tau_{\beta\gamma}}{h_1h_2}\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\} \\
Z &= \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{p_\gamma}{h_1h_2}\,d\gamma + \left|\frac{\sigma_\gamma}{h_1h_2}\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\}
\end{aligned}\right\} \tag{5.5}$$

The magnitudes $m_\alpha = m_\alpha(\alpha,\beta)$ and $m_\beta = m_\beta(\alpha,\beta)$ are the moments of all the forces (surface and volume) relative to the axes α,β of the movable trihedron of the middle surface:

$$\left.\begin{aligned}
m_\alpha &= \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{p_\beta}{h_1h_2}\,\gamma d\gamma + \left|\frac{\tau_{\beta\gamma}}{h_1h_2}\,\gamma\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\} \\
m_\beta &= \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{p_\alpha}{h_1h_2}\,\gamma d\gamma + \left|\frac{\tau_{\alpha\gamma}}{h_1h_2}\,\gamma\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\}
\end{aligned}\right\} \tag{5.6}$$

Finally the magnitude $Z^* = Z^*(\alpha,\beta)$ is the new generalized static magnitude corresponding to the elongation of the normal element and determined by the equation

$$Z^* = \frac{1}{AB}\left\{\int_{-\frac{1}{2}\delta}^{+\delta} \frac{p_\gamma}{h_1h_2}\,\gamma d\gamma + \left|\frac{\sigma_\gamma \gamma}{h_1h_2}\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta}\right\} \tag{5.7}$$

In the case where only surface forces act on the shell, the first components in equations (5.5) to (5.7) drop out.

To the equilibrium equations (5.4) must be added the equations for the components Δ_0, Δ_1, and Δ_2 of the volume dilation and χ_0, χ_1, and χ_2 of the normal rotation. According to equations (3.9),

$$\left.\begin{aligned}
\Delta_0 &= \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu) + \frac{\partial}{\partial\beta}(Av)\right] + 2Hw + w^* \\
\Delta_1 &= 2\left(\frac{\partial H}{\partial\alpha}\frac{u}{A} + \frac{\partial H}{\partial\beta}\frac{v}{B}\right) - \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial w}{\partial\beta}\right)\right] - (k^2{}_1+k^2{}_2)w + 2Hw^* \\
\Delta_2 &= -\frac{1}{2}\left[\frac{\partial}{\partial\alpha}(k^2{}_1+k^2{}_2)\frac{u}{A} + \frac{\partial}{\partial\beta}(k^2{}_1+k^2{}_2)\frac{v}{B}\right] + k_1\left[\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}\right] + \\
&\quad k_2\left[\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}\right] + (k^3{}_1+k^3{}_2)w - (k^2{}_1+k^2{}_2)w^* \\
\chi_0 &= \frac{1}{2AB}\left[\frac{\partial}{\partial\alpha}(Bv) - \frac{\partial}{\partial\beta}(Au)\right] \\
\chi_1 &= -L\left[\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right)\right] \\
\chi_2 &= L\left[k_1\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + k_2\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + \frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right]
\end{aligned}\right\} \tag{5.8}$$

In equations (5.4) and (5.8),

$$\left.\begin{aligned}
K &= k_1k_2 \\
H &= \frac{1}{2}(k_1+k_2) \\
L &= \frac{1}{2}(k_1-k_2)
\end{aligned}\right\} \tag{5.9}$$

Equations (5.4) together with equations (5.8) form a complete system of differential equations of the shell. To these equations the boundary conditions for each particular case must be added.

For this purpose, the internal generalized forces must be determined. These forces, corresponding to the degrees of freedom of the normal deformable element, will in any normal sections of the shell consist of the tangential (normal and shearing) forces T and S acting in the plane tangent to $\gamma = 0$ and corresponding to the displacements of the element parallel to this plane, the transverse force N directed along the normal to $\gamma = 0$ and corresponding to the displacement of the element along the normal to the middle surface $\gamma = 0$, the bending and torsional moments G and H corresponding to the angular displacements of the element with respect to the tangent axes of the movable trihedron and, finally, the new generalized (statically equivalent to zero) transverse force N^* corresponding to the elongation of the normal element.

All of these forces, with the exception of the transverse force N, can be expressed in terms of the fundamental kinematic magnitudes u, v, w, and w^* by setting up the work of all (tangential and normal) stresses of the normal section considered over unit displacements of the normal element, translational in the tangent plane, translational in the direction of the normal to the middle surface, angular relative to the axes in the tangent plane, and in the displacement of the points of the element $u_\gamma = w^*\gamma$ for $w^* = 1$.

For the internal forces on the two basic normal sections $\alpha = \text{constant}$ and $\beta = \text{constant}$ (fig. 4), the following equations are obtained:

$$
\left.
\begin{aligned}
T_1 &= \frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\sigma_\alpha}{h_2}\,d\gamma = (\lambda+2\mu)\delta\left[\epsilon_1 + \frac{\delta^2}{12}(k_2-k_1)\chi_1\right] + \lambda\delta(\epsilon_2+w^*) \\
T_2 &= \frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\sigma_\beta}{h_1}\,d\gamma = (\lambda+2\mu)\delta\left[\epsilon_2 + \frac{\delta^2}{12}(k_1-k_2)\chi_2\right] + \lambda\delta(\epsilon_1+w^*) \\
S_1 &= \frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\beta\alpha}}{h_2}\,d\gamma = \mu\delta\left[\omega + \frac{\delta^2}{12}(k_2\tau_1+\tau_2)\right] \\
S_2 &= -\frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\alpha\beta}}{h_1}\,d\gamma = -\mu\delta\left[\omega + \frac{\delta^2}{12}(k_1\tau_1+\tau_2)\right] \\
G_1 &= -\frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\sigma_\alpha}{h_2}\,\gamma d\gamma = -\frac{\delta^3}{12}\left[(\lambda+2\mu)(\chi_1+k_2\epsilon_1) + \lambda(\chi_2+k_2\epsilon_2+k_2w^*)\right] \\
G_2 &= -\frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\sigma_\beta}{h_1}\,\gamma d\gamma = -\frac{\delta^3}{12}\left[(\lambda+2\mu)(\chi_2+k_1\epsilon_2) + \lambda(\chi_1+k_1\epsilon_1+k_1w^*)\right] \\
H_1 &= \frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\beta\alpha}}{h_2}\,\gamma d\gamma = \frac{\mu\delta^3}{12}(\tau_1+k_2\omega) \\
H_2 &= -\frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\alpha\beta}}{h_1}\,\gamma d\gamma = -\frac{\mu\delta^3}{12}(\tau_1+k_1\omega) \\
N_1^* &= \frac{1}{B}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\gamma\alpha}}{h_2}\,\gamma d\gamma = \frac{\mu\delta^3}{12}\,\frac{1}{A}\,\frac{\partial w^*}{\partial\alpha} \\
N_2^* &= \frac{1}{A}\int_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \frac{\tau_{\gamma\beta}}{h_1}\,\gamma d\gamma = \frac{\mu\delta^3}{12}\,\frac{1}{B}\,\frac{\partial w^*}{\partial\beta}
\end{aligned}
\right\}
\quad (5.10
$$

In these equations,

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_1 w \\
\epsilon_2 &= \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + \frac{1}{B}\frac{\partial v}{\partial \beta} + k_2 w \\
\omega &= \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) \\
\chi_1 &= \frac{\partial k_1}{\partial \alpha}\frac{u}{A} + \frac{\partial k_1}{\partial \beta}\frac{v}{B} - k_1^2 w - \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + k_1 w^* \\
\chi_2 &= \frac{\partial k_2}{\partial \alpha}\frac{u}{A} + \frac{\partial k_2}{\partial \beta}\frac{v}{B} - k_2^2 w - \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2 B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + k_2 w^* \\
\tau_1 &= (k_1 - k_2)\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right] - \frac{2}{AB}\left(\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha}\right) \\
\tau_2 &= (k_1 - k_2)\left[k_2 \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - k_1 \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right] + \frac{k_1 + k_2}{AB}\left(\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha}\right)
\end{aligned}\right\} \tag{5.11}$$

Depending on the character of the problem, the boundary conditions may be purely kinematic, purely static, or of the mixed type.

In the case of kinematic conditions for the normal element of the shell and the boundary surface, there must be given in the boundary surface three displacements of the midpoint of the element along three mutually perpendicular directions, the angle of rotation of the element relative to the tangent to the contour curve of the middle surface and finally the normal displacement of any other point of the element. Altogether there will be five independent kinematic conditions, which together with the fundamental equations (5.2) and (5.8) make the problem entirely determinate.

If the boundary conditions are given in terms of stresses, there will be in this case five independent conditions, the four usual conditions of the moment theory and one with regard to the generalized (statically equivalent to zero) transverse force N^*.

6. Shells of Medium Thickness. - By neglecting in equations (5.4) and (5.8) the small terms with tangential displacements u and v, which contain as factors the products of the curvatures k_1 and k_2, and their derivatives for the shell of medium thickness

$$\left.\begin{aligned} w^* &= 0 \\ \lambda &= \frac{E\nu}{1-\nu^2} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned}\right\} \qquad (6.1)$$

and neglecting, in correspondence with this the last equation of (5.4)

$$\begin{aligned}
&\frac{1}{A}\left(\frac{\partial\Delta_0}{\partial\alpha} + \frac{\delta^2}{6} H \frac{\partial\Delta_1}{\partial\alpha} + \frac{\delta^2}{12}\frac{\partial\Delta_2}{\partial\alpha}\right) - (1-\nu)\frac{1}{B}\left(\frac{\partial\chi_0}{\partial\beta} + \frac{\delta^2}{6} k_1 \frac{\partial\chi_1}{\partial\beta} + \frac{\delta^2}{12}\frac{\partial\chi_2}{\partial\beta}\right) + \\
&\qquad (1+\nu)\left(Ku - \frac{k_2}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{1-\nu^2}{E\delta}(X - k_1 m_\beta) = 0 \\
&\frac{1}{B}\left(\frac{\partial\Delta_0}{\partial\beta} + \frac{\delta^2}{6} H \frac{\partial\Delta_1}{\partial\beta} + \frac{\delta^2}{12}\frac{\partial\Delta_2}{\partial\beta}\right) + (1-\nu)\frac{1}{A}\left(\frac{\partial\chi_0}{\partial\alpha} + \frac{\delta^2}{6} k_2 \frac{\partial\chi_1}{\partial\alpha} + \frac{\delta^2}{12}\frac{\partial\chi_2}{\partial\alpha}\right) + \\
&\qquad (1-\nu)\left(Kv - \frac{k_1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1-\nu^2}{E\delta}(Y + k_2 m_\alpha) = 0 \\
&\frac{\delta^2}{12}\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\left[\frac{B}{A}\left(k_2\frac{\partial\Delta_0}{\partial\alpha} + \frac{\partial\Delta_1}{\partial\alpha} - (1-\nu)K\frac{\partial w}{\partial\alpha}\right) - (1-\nu)k_1\frac{\partial\chi_0}{\partial\beta}\right] + \right. \\
&\qquad \left.\frac{\partial}{\partial\beta}\left[\frac{A}{B}\left(k_1\frac{\partial\Delta_0}{\partial\beta} + \frac{\partial\Delta_1}{\partial\beta} - (1-\nu)K\frac{\partial w}{\partial\beta}\right) + (1-\nu)k_2\frac{\partial\chi_0}{\partial\alpha}\right]\right\} - \\
&\frac{\delta^2}{6}(K\Delta_1 + H\Delta_2) - 2H\Delta_0 + (1-\nu)\frac{1}{AB}\left[2ABKw + \frac{\partial}{\partial\alpha}(Bk_2u) + \frac{\partial}{\partial\beta}(Ak_1v)\right] + \\
&\qquad \frac{1-\nu^2}{E\delta}\left\{Z - \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bm_\beta) - \frac{\partial}{\partial\beta}(Am_\alpha)\right]\right\} = 0
\end{aligned} \qquad (6.2)$$

where now

$$
\left.
\begin{aligned}
\Delta_0 &= \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu) + \frac{\partial}{\partial\beta}(Av)\right] + 2Hw \\
\Delta_1 &= 2\left(\frac{\partial H}{\partial\alpha}\frac{u}{A} + \frac{\partial H}{\partial\beta}\frac{v}{B}\right) - (k^2{}_1 + k^2{}_2)w - \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial w}{\partial\beta}\right)\right] \\
\Delta_2 &= \frac{H}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial w}{\partial\beta}\right)\right] + \frac{L}{AB}\left[B\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - A\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \right. \\
&\qquad \left. \frac{1}{B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} - \frac{1}{A}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}\right] \\
{}_0 &= \frac{1}{2AB}\left[\frac{\partial}{\partial\alpha}(Bv) - \frac{\partial}{\partial\beta}(Au)\right] \\
{}_1 &= -L\left[\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right)\right] \\
{}_2 &= L\left(\frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)
\end{aligned}
\right\}
\tag{6.3}
$$

For the internal tangential forces and the moments,

$$
\left.
\begin{aligned}
T_1 &= \frac{E\delta}{1-\nu^2}\left[\epsilon_1 + \nu\epsilon_2 - \frac{\delta^2}{12}(k_1 - k_2)\chi_1\right] \\
T_2 &= \frac{E\delta}{1-\nu^2}\left[\epsilon_2 + \nu\epsilon_1 + \frac{\delta^2}{12}(k_1 - k_2)\chi_2\right] \\
S_1 &= \frac{E\delta}{2(1+\nu)}\left[\omega + \frac{\delta^2}{12}(k_2\tau_1 + \tau_2)\right] \\
S_2 &= -\frac{E\delta}{2(1+\nu)}\left[\omega + \frac{\delta^2}{12}(k_1\tau_1 + \tau_2)\right] \\
G_1 &= -\frac{E\delta^3}{12(1-\nu^2)}\left[\chi_1 + k_2\epsilon_1 + \nu(\chi_2 + k_3\epsilon_2)\right] \\
G_2 &= -\frac{E\delta^3}{12(1-\nu^2)}\left[\chi_2 + k_1\epsilon_2 + \nu(\chi_1 + k_1\epsilon_1)\right] \\
H_1 &= \frac{E\delta^3}{24(1+\nu)}(\tau_1 + k_2\omega) \\
H_2 &= -\frac{E\delta^3}{24(1+\nu)}(\tau_1 + k_1\omega)
\end{aligned}
\right\}
\tag{6.4}
$$

where the strains ϵ_1, ϵ_2, ω_1, χ_1, χ_2, τ_1, and τ_2 are determined by equations (5.11), in which (fourth and fifth) w^* must be set equal to zero.

Equations (6.2) to (6.4) refer to shells of medium thickness and were obtained with an accuracy up to terms with $\delta^3/12$ in strict correspondence with the fundamental assumption (3.2) because, in the first place, the given equations (4.1) for the components of the changes in curvature χ_1, χ_2, and τ_1, in contrast to equations (4.6) of Love are accurate; and in the second place, these equations contain additional terms arising from the moments, namely, the terms with $\Delta_2\delta^3/12$, and $\chi_2\delta^3/12$, which are absent in the present moment theory constructed on the basis of the magnitudes $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ in the form of equations (4.5) and not in the form

$$\left.\begin{aligned} e_{\alpha\alpha} &= \epsilon_1 + \chi_{11}\gamma + \chi_{12}\gamma^2 \\ e_{\beta\beta} &= \epsilon_2 + \chi_{21}\gamma + \chi_{22}\gamma^2 \\ e_{\alpha\beta} &= \omega + \tau_1\gamma + \tau_2\gamma^2 \end{aligned}\right\} \qquad (6.5)$$

which lie at the basis of the theory given here. This theory and the more general one given in the preceding section and referring to thicker shells is in full agreement with the fundamental theorems of the theory of elasticity, in particular, with the theorem of reciprocal work, which, as shown later for the example of a cylindrical shell, does not correspond to the theory of Love.

7. General Technical Theory of Thin Shells. Two methods of Solution of the Problem. Generalization of the Maxwell and Sophie Germain-Lagrange Equations. - For thin shells, further simplification of equations (6.2) to (6.4) is possible. Eliminating from equations (6.2) the functions Δ_0, Δ_1, Δ_2, χ_0, χ_1, and χ_2 with the aid of relations (6.3), three equations are obtained in the three functions u, v, and w. The first two of these equations, when multiplied by δ, will each consist of terms proportional to the thickness δ and terms proportional to magnitudes consisting of the product of $\delta^3/12$ by the curvatures k_1 and k_2 or the derivatives of these curvatures.

In the third equation, in addition to the terms of this type, there will enter a term proportional to $\delta^3/12$ and independent of the curvature of the shell.

The rather extensive theoretical and experimental investigations made by the author show that for thin shells the relative thickness $\delta k_{max} = \delta/R_{min} < 1/30$; the terms with $\frac{\delta^3}{12} k_1$, $\frac{\delta^3}{12} k_2$, $\frac{\delta^3}{12} \frac{\partial k_1}{\partial \alpha}$, $\frac{\delta^3}{12} \frac{\partial k_3}{\partial \alpha}$, $\frac{\delta^3}{12} \frac{\partial k_1}{\partial \beta}$, and $\frac{\delta^3}{12} \frac{\partial k_2}{\partial \beta}$ entering in the fundamental equations are factors of second-order values for the displacements u and v. Without sensible error these terms, as shown in the work on cylindrical shells (reference 7) and thin-walled rods (reference 8), can be neglected. Correspondingly,

$$\left.\begin{aligned}
T_1 &= \frac{E\delta}{1-\nu^2} (\epsilon_1+\nu\epsilon_2) \\
T_2 &= \frac{E\delta}{1-\nu^2} (\epsilon_2+\nu\epsilon_1) \\
G_1 &= - \frac{E\delta^3}{12(1-\nu_2)} (\chi_1+\nu\chi_2) \\
G_2 &= - \frac{E\delta^3}{12(1-\nu^2)} (\chi_2+\nu\chi_1) \\
S_1 &= - S_2 = \frac{E\delta}{2(1+\nu)} \omega \\
H_1 &= - H_2 = \frac{E\delta^3}{12(1+\nu)} \sigma
\end{aligned}\right\} \tag{7.1}$$

where because of the assumptions made,

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_1 w \\
\epsilon_2 &= \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + \frac{1}{B}\frac{\partial v}{\partial \beta} + k_2 w \\
\omega &= \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{\beta}\right) \\
\chi_1 &= -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} \\
\chi_2 &= -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2 B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} \\
\tau &= -\frac{1}{AB}\left(\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha}\right)
\end{aligned}\right\} \qquad (7.2)$$

Neglecting in equations (6.2) all terms with $\delta^2/12$ except for the term with $\frac{\delta^2}{12}\frac{1}{AB}\left[\frac{\partial}{\partial \alpha}\left(\frac{B}{A}\frac{\partial \Delta_1}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{A}{B}\frac{\partial \Delta_1}{\partial \beta}\right)\right]$ in the last equation, giving according to equations (5.8) a value independent of the curvature of the shell introducing the new functions $\varphi = \varphi(\alpha,\beta)$ and $\psi = \psi(\alpha,\beta)$

$$\left.\begin{aligned}
u &= \frac{1}{A}\frac{\partial \varphi}{\partial \alpha} + \frac{1}{B}\frac{\partial \psi}{\partial \beta} \\
v &= \frac{1}{B}\frac{\partial \varphi}{\partial \beta} - \frac{1}{A}\frac{\partial \psi}{\partial \alpha}
\end{aligned}\right\} \qquad (7.3)$$

After certain transformations and simplifications of equations (6.2),

$$\left.\begin{aligned}
&\nabla_e^2\nabla_e^2\varphi + 2\nabla_e^2(Hw) - (1-\nu)(H\nabla_e^2 - L\nabla_h^2)w = -\frac{1-\nu^2}{E\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right]\\
&\frac{1-\nu}{2}\nabla_e^2\nabla_e^2\psi + (1-\nu)\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(k_1\frac{\partial w}{\partial\beta}\right) - \frac{\partial}{\partial\beta}\left(k_2\frac{\partial w}{\partial\alpha}\right)\right]\\
&\qquad = -\frac{1-\nu^2}{E\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AX) - \frac{\partial}{\partial\alpha}(BY)\right]\\
&-2H\nabla_e^2\varphi + (1-\nu)(H\nabla_e^2 - L\nabla_h^2)\varphi + (1-\nu)\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(k_2\frac{\partial\psi}{\partial\beta}\right) - \frac{\partial}{\partial\beta}\left(k_1\frac{\partial\psi}{\partial\alpha}\right)\right] -\\
&\qquad \frac{\delta^2}{12}\nabla_e^2\nabla_e^2 w - 2\left[2H^2 - (1-\nu)K\right]w = -\frac{1-\nu^2}{E\delta}Z
\end{aligned}\right\} \tag{7.4}$$

where φ, ψ, and w are the required functions of the displacements and are invariant (relative to the directions of the coordinate curves α and β at a given point of the surface) magnitudes; ∇_e^2 and ∇_h^2 are the differential operators of the second order of the elliptic and hyperbolic type:

$$\left.\begin{aligned}
&\nabla_e^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right] =\\
&\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\right) + \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\\
&\nabla_h^2 = \frac{1}{AB}\left[B^2\frac{\partial}{\partial\alpha}\left(\frac{1}{AB}\frac{\partial}{\partial\alpha}\right) - A^2\frac{\partial}{\partial\beta}\left(\frac{1}{AB}\frac{\partial}{\partial\beta}\right)\right] =\\
&\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\right) - \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}
\end{aligned}\right\} \tag{7.5}$$

The mixed operator $H\nabla_e^2 - L\nabla_h^2$ is defined by the equation

$$\left.\begin{gathered}H\nabla_e^{\ 2} - L\nabla_h^{\ 2} = \frac{k_1+k_2}{2}\,\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\,\frac{\partial}{\partial\beta}\right)\right] - \\ \frac{k_1-k_2}{2}\,\frac{1}{AB}\left[B^2\,\frac{\partial}{\partial\alpha}\left(\frac{1}{AB}\,\frac{\partial}{\partial\alpha}\right) - A^2\,\frac{\partial}{\partial\beta}\left(\frac{1}{AB}\,\frac{\partial}{\partial\beta}\right)\right] = \\ k_1\left[\frac{1}{B}\,\frac{\partial}{\partial\beta}\left(\frac{1}{B}\,\frac{\partial}{\partial\beta}\right) + \frac{1}{A^2B}\,\frac{\partial B}{\partial\alpha}\,\frac{\partial}{\partial\alpha}\right] + k_2\left[\frac{1}{A}\,\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\,\frac{\partial}{\partial\alpha}\right) + \frac{1}{AB^2}\,\frac{\partial A}{\partial\beta}\,\frac{\partial}{\partial\beta}\right] = \\ \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\,k_2\,\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\,k_1\,\frac{\partial}{\partial\beta}\right)\right]\end{gathered}\right\} \tag{7.6}$$

Differential equations (7.4) form a complete system of equations in the three fundamental displacement functions φ, ψ, and w.

These functions according to equations (7.3) determine the vector of total displacement of the point (α,β) of the middle surface and therefore by virtue of equations (7.2) and (7.1) all the deformations and the internal axial forces and moments of the shell.

Equations (7.4) are thus the fundamental equations of the theory given here for thin shells and permit solution of the problem of the equilibrium of elastic shells of small curvature by the method of displacements.

The theory of thin shells can also be presented in another more compact form, namely, in the form earlier proposed of the mixed method by introducing only two functions, the stress function Φ and the displacement function w. Setting (for $X = Y = 0$)

$$\left.\begin{aligned}T_1 &= \frac{1}{B}\,\frac{\partial}{\partial\beta}\left(\frac{1}{B}\,\frac{\partial\Phi}{\partial\beta}\right) + \frac{1}{A^2B}\,\frac{\partial B}{\partial\alpha}\,\frac{\partial\Phi}{\partial\alpha} \\ T_2 &= \frac{1}{A}\,\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\,\frac{\partial\Phi}{\partial\alpha}\right) + \frac{1}{AB^2}\,\frac{\partial A}{\partial\beta}\,\frac{\partial\Phi}{\partial\beta} \\ S_1 &= -\,S_2 = -\,\frac{1}{AB}\left(\frac{\partial^2\Phi}{\partial\alpha\partial\beta} - \frac{1}{A}\,\frac{\partial B}{\partial\alpha}\,\frac{\partial\Phi}{\partial\beta} - \frac{1}{B}\,\frac{\partial A}{\partial\beta}\,\frac{\partial\Phi}{\partial\alpha}\right)\end{aligned}\right\} \tag{7.7}$$

and bearing in mind the analogous equations (7.2) for X_1, X_2, and π and equations (7.1) for G_1, G_2, and $H_1 = -H_2$, the general equations of equilibrium and deformations of the moment theory of shells are represented in the form of two symmetrically constructed differential equations:

$$\left.\begin{aligned} \frac{1}{E\delta}\nabla_e^2\nabla_e^2\Phi - (H\nabla_e^2 - L\nabla_h^2)w = 0 \\ -(H\nabla_e^2 - L\nabla_h^2)\Phi - \frac{E\delta^3}{12(1-\nu^2)}\nabla_e^2\nabla_e^2 w + Z = 0 \end{aligned}\right\} \quad (7.8)$$

These equations are a generalization of the equation of Maxwell for the two-dimensional stress state of a plate and the equation of Sophie Germain - Lagrange for the case of the bending of a plate, inasmuch as for $k_1 = k_2 = 0$ (the case of a flat plate) they break down into the well-known equations from the theory of elasticity

$$\left.\begin{aligned} \nabla_e^2\nabla_e^2\Phi = 0 \\ \nabla_e^2\nabla_e^2 w = \frac{12(1-\nu^2)}{E\delta^3} Z \end{aligned}\right\} \quad (7.9)$$

in arbitrary (for ∇_e^2 determined by equations (7.5)) orthogonal coordinates.

If in the second of equations (7.8) the term with $\delta^3/12$ is neglected, the fundamental equation of the momentless theory of shells results:

$$(H\nabla_e^2 - L\nabla_h^2)\Phi = Z \quad (7.10)$$

After determining the functions Φ and w, the forces T_1 and T_2 are found from equations (7.7), the moments G_1, G_2, and H from equations (7.2) and (7.1).

These forces and moments will satisfy the equations of equilibrium

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha}(BT_1) - T_2\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(AS_2) + S_1\frac{\partial A}{\partial\beta} + ABk_1N_1 + ABX = 0\\
&\frac{\partial}{\partial\beta}(AT_2) - T_1\frac{\partial A}{\partial\beta} + \frac{\partial}{\partial\alpha}(BS_1) - S_2\frac{\partial B}{\partial\alpha} + ABk_2N_2 + ABY = 0\\
&-(k_1T_1+k_2T_2) + \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BN_1) + \frac{\partial}{\partial\beta}(AN_2)\right] + Z = 0\\
&\frac{\partial}{\partial\alpha}(BH_1) - H_2\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(AG_2) + G_1\frac{\partial A}{\partial\beta} - ABN_2 = 0\\
&\frac{\partial}{\partial\beta}(AH_2) - H_1\frac{\partial A}{\partial\beta} + \frac{\partial}{\partial\alpha}(BG_1) - G_2\frac{\partial B}{\partial\alpha} + ABN_1 = 0\\
&S_1 + S_2 + k_1H_1 + k_2H_2 = 0
\end{aligned}\right\} \tag{7.11}$$

for $X = Y = 0$ with an accuracy up to the terms ABk_1N_1 and ABk_2N_2 in the first two equations and the term $k_1H_1 + k_2H_2$ in the last equation, which as magnitudes proportional to $k_1\delta^3/12$ and $k_2\delta^3/12$ (by virtue of the fourth and fifth equations of (7.11) and the relations (7.1) and (7.2) for the moments) are taken equal to zero.

In neglecting in the first two equations the terms with k_1N_1 and k_2N_2, an error is admitted of the same order as that in the general theory in replacing the last of equations (7.11) by the approximate relation $S_1 = - S_2$.

For the transverse forces N_1 and N_2,

$$\left.\begin{aligned}
N_1 &= -\frac{E\delta^3}{12(1-\nu^2)}\frac{1}{A}\frac{\partial}{\partial\alpha}\nabla_e{}^2 w\\
N_2 &= -\frac{E\delta^3}{12(1-\nu^2)}\frac{1}{B}\frac{\partial}{\partial\beta}\nabla_e{}^2 w
\end{aligned}\right\} \tag{7.12}$$

which constitute a generalization of the well-known equations of the theory of the bending of a plate.

8. Circular Cylindrical Shell. Particular Cases. - For the coordinates of α and β, the distance to the point considered along the generator and the transverse arc, respectively, of the middle surface are taken.

Then, evidently $A = B = 1$. Equations (5.8) for $k_1 = 0$ and $k_2 = 1/R =$ constant assume the form

$$\left.\begin{aligned}
\Delta_0 &= \frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \beta} + k_2 w + w^* \\
\Delta_1 &= -\left(\frac{\partial^2 w}{\partial \alpha^2} + \frac{\partial^2 w}{\partial \beta^2} + k^2{}_2 w\right) + k_2 w^* \\
\Delta_2 &= k_1 \\
\chi_0 &= \frac{1}{2}\left(\frac{\partial v}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right) \\
\chi_1 &= + \frac{k_2}{2}\left(\frac{\partial v}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\
\chi_2 &= - \frac{k_2}{2}\left(\frac{\partial^2 w}{\partial \alpha \partial \beta} + k_2 \frac{\partial u}{\partial \beta}\right)
\end{aligned}\right\} \tag{8.1}$$

Equations (5.4) may be presented with the aid of table 1, containing the differential operators.

TABLE 1

$u(\alpha,\beta)$	$v(\alpha,\beta)$	$w(\alpha,\beta)$	$w^*(\alpha,\beta)$	
$(\lambda+2\mu)\frac{\partial^2}{\partial\alpha^2}+\mu\frac{\partial^2}{\partial\beta^2}$	$(\lambda+\mu)\frac{\partial^2}{\partial\alpha\partial\beta}$	$\lambda k_2\frac{\partial}{\partial\alpha}-(\lambda+2\mu)\frac{\delta^2}{12}k_2\frac{\partial^3}{\partial\alpha^3}+\mu\frac{\delta^2}{12}k_2\frac{\partial^3}{\partial\alpha\partial\beta^2}$	$\lambda\frac{\partial}{\partial\alpha}$	$\frac{1}{\delta}X$
$(\lambda+\mu)\frac{\partial^2}{\partial\alpha\partial\beta}$	$(\lambda+2\mu)\frac{\partial^2}{\partial\beta^2}+\mu\frac{\partial^2}{\partial\alpha^2}$	$(\lambda+2\mu)k_2\frac{\partial}{\partial\beta}-(\lambda+3\mu)k_2\frac{\delta^2}{12}\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$\lambda\frac{\partial}{\partial\beta}$	$\frac{1}{\delta}Y$
$\lambda k_2\frac{\partial}{\partial\alpha}-(\lambda+2\mu)\frac{\delta^2}{12}k_2\frac{\partial^3}{\partial\alpha^3}+\mu\frac{\delta^2}{12}k_2\frac{\partial^3}{\partial\alpha\partial\beta^2}$	$(\lambda+2\mu)k_2\frac{\partial}{\partial\beta}-(\lambda+3\mu)k_2\frac{\delta^2}{12}\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$(\lambda+2\mu)\left[k_2^2+\frac{\delta^2}{12}\left(k_2^4+2k_2^2\frac{\partial^2}{\partial\beta^2}+\nabla^2\nabla^2\right)\right]$	$\lambda k_2-\frac{\delta^2}{12}k_2\left[2\lambda\frac{\partial^2}{\partial\alpha^2}+(\lambda+2\mu)\left(\frac{\partial^2}{\partial\beta^2}+k_2^2\right)\right]$	$-\frac{1}{\delta}Z$
$\lambda\frac{\partial}{\partial\alpha}$	$\lambda\frac{\partial}{\partial\beta}$	$\lambda k_2-\frac{\delta^2}{12}k_2\left[2\lambda\frac{\partial^2}{\partial\alpha^2}+(\lambda+2\mu)\left(\frac{\partial^2}{\partial\beta^2}+k_2^2\right)\right]$	$(\lambda+2\mu)-\mu\frac{\delta^2}{12}\nabla^2$	$-\frac{1}{\delta}Z^*$

(8.2)

In this table X, Y, Z, and Z* denote free terms depending on the internal forces (m_α and m_β are taken equal to zero). The differential operators referring to the corresponding functions are indicated. These operators form a symmetrical differential matrix. The elements of the matrix symmetrical with respect to the diagonal terms have the same expressions, a consequence of the theorem of reciprocal work.

For $k_2 = 0$, equations (8.2) break down into the following equations (likewise symmetrically constructed):

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial^2 u}{\partial\alpha^2} + \mu\frac{\partial^2 u}{\partial\beta^2} + (\lambda+\mu)\frac{\partial^2 v}{\partial\alpha\partial\beta} + \lambda\frac{\partial w^*}{\partial\alpha} + \frac{1}{\delta}X = 0\\
&(\lambda+\mu)\frac{\partial^2 u}{\partial\alpha\partial\beta} + (\lambda+2\mu)\frac{\partial^2 v}{\partial\beta^2} + \mu\frac{\partial^2 v}{\partial\alpha^2} + \lambda\frac{\partial w^*}{\partial\beta} + \frac{1}{\delta}Y = 0\\
&\lambda\frac{\partial u}{\partial\alpha} + \lambda\frac{\partial u}{\partial\beta} - \mu\nabla^2 w^* + (\lambda+2\mu)w^* - \frac{1}{\delta}Z = 0
\end{aligned}\right\} \tag{8.3}$$

and the equation for the bending of a plate

$$\nabla^2\nabla^2 w = \frac{12}{(\lambda+2\mu)\delta^3}Z \tag{8.4}$$

Equations (8.3) are, in a certain sense, a generalization of the problem of the two-dimensional stress state of a plate and permit determining the stresses and the deformations of the plate under the action of two mutually balancing concentrated forces applied on the planes $\gamma = +\frac{1}{2}\delta$ and $\gamma = -\frac{1}{2}\delta$ and acting normal to the middle surface. In the case of the homogeneous problem, the first two equations of (8.3) may be satisfied by setting

$$\left.\begin{aligned}
u &= -\lambda\frac{\partial}{\partial\alpha}\nabla^2\Phi\\
v &= -\lambda\frac{\partial}{\partial\beta}\nabla^2\Phi\\
w^* &= (\lambda+2\mu)\nabla^2\nabla^2\Phi
\end{aligned}\right\} \tag{8.5}$$

where $\Phi = \Phi(\alpha,\beta)$ is an arbitrary function. The last equation then becomes

$$\nabla^2\nabla^2\nabla^2\Phi - \frac{4(\lambda+\mu)}{\lambda+2\mu}\nabla^2\nabla^2\Phi = 0 \tag{8.6}$$

and therefore the function $\nabla^2\Phi - 4(\lambda+\mu)/(\lambda+2\mu)$ is biharmonic.

The magnitudes u, v, and w^* determine the strains $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$, and $e_{\gamma\gamma}$ and therefore the stresses σ_α, σ_β, and $\tau_{\alpha\beta}$. The remaining stresses, as in the general case of the shell, must be found from the condition of equilibrium.

In the same manner as the particular case of equations (8.2), there can be obtained the fundamental equations for the circular arch with account taken of the extensional deformations of the arch in the direction of the normal to its axis. In this case, the displacement v must be considered equal to zero and the remaining magnitudes u, w, and w^* considered only as functions of β. Equations, which generalize the well-known equations of Boussinesq, are obtained.

For

$$\left.\begin{aligned} w^* &= 0 \\ \lambda &= \frac{E\nu}{1-\nu^2} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned}\right\}$$

(where ν is the Poisson coefficient), equations for the circular cylindrical shell shall be obtained in the three functions u, v, and w. The last of equations (6.2) drops out and the remaining ones, in passing to the relative coordinates so that $A = B = R$, may be represented with the aid of table 2.

TABLE 2

$u(\alpha,\beta)$	$v(\alpha,\beta)$	$w(\alpha,\beta)$	
$\frac{\partial^2}{\partial\alpha^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2}$	$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\nu\frac{\partial}{\partial\alpha} - c^2\left(\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right)$	$\frac{1-\nu^2}{E\delta}R^2X$
$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\frac{\partial^2}{\partial\beta^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2}$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}c^2\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$\frac{1-\nu^2}{E\delta}R^2Y$
$\nu\frac{\partial}{\partial\alpha} - c^2\left(\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right)$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}c^2\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$c^2\left(\nabla^2\nabla^2 + 2\frac{\partial^2}{\partial\beta^2} + 1\right) + 1$	$-\frac{1-\nu^2}{R\delta}R^2Z$

(8.7)

where

$$\left.\begin{aligned} c^2 &= \frac{\delta^2}{12R^2} \\ \nabla^2 &= \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} \end{aligned}\right\} \qquad (8.8)$$

The system of the three equations (8.7) may be reduced to an equivalent single equation of the eighth order. Following Galerkin (references 7 and 8), the first two equations (8.7) may be substituted for $X = Y = 0$ by introducing a new function $\Phi = \Phi(\alpha,\beta)$ and expressing in terms of this function the displacements u, v, and w by the equations

$$\left.\begin{aligned} u &= c^2\left(\frac{\partial^5\Phi}{\partial\alpha^5} - \frac{\partial^5\Phi}{\partial\alpha\partial\beta^4}\right) + \frac{\partial^3\Phi}{\partial\alpha\partial\beta^2} - \nu\,\frac{\partial^3\Phi}{\partial\alpha^3} \\ v &= 2c^2\left(\frac{\partial^5\Phi}{\partial\alpha^4\partial\beta} + \frac{\partial^5\Phi}{\partial\alpha^2\partial\beta^3}\right) - (2+\nu)\,\frac{\partial^3\Phi}{\partial\alpha^2\partial\beta} - \frac{\partial^3\Phi}{\partial\beta^3} \\ w &= \frac{\partial^4\Phi}{\partial\alpha^4} + 2\,\frac{\partial^4\Phi}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4\Phi}{\partial\beta^4} \end{aligned}\right\} \tag{8.9}$$

The last of equations (8.7) assumes the form

$$\left.\begin{aligned} c^2(\nabla^2\nabla^2 + 2\nabla^2 + 1)\nabla^2\nabla^2\Phi - 2c^2(1-\nu)\left(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\nabla^2\Phi + \\ (1-\nu^2)\,\frac{\partial^4\Phi}{\partial\alpha^4} - \frac{(1-\nu^2)\delta}{12\,Ec^2}\,Z = 0 \end{aligned}\right\} \tag{8.10}$$

Equation (8.10) is the fundamental equation of the circular shell. In this equation

$$\nabla^2\nabla^2 = \frac{\partial^4}{\partial\alpha^4} + 2\,\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} \tag{8.11}$$

For comparison, there are presented the equations obtained on the basis of the existing moment theory of Love. These equations, given for example in the book of Timoshenko, may also be represented with the aid of table 3.

TABLE 3

$u(\alpha,\beta)$	$v(\alpha,\beta)$	$w(\alpha,\beta)$	
$\frac{\partial^2}{\partial\alpha^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2}$	$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\nu\frac{\partial}{\partial\alpha}$	$\frac{1-\nu^2}{E\delta}R^2X$
$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\frac{\partial^2}{\partial\beta^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2}$	$\frac{\partial}{\partial\beta} - c^2\left(\frac{\partial^3}{\partial\beta^3} + \frac{\partial^2}{\partial\alpha^2\partial\beta}\right)$	$\frac{1-\nu^2}{E\delta}R^2Y$
$\nu\frac{\partial}{\partial\alpha}$	$\frac{\partial}{\partial\beta} - c^2\left[\frac{\partial^3}{\partial\beta^3} + (2-\nu)\frac{\partial^3}{\partial\alpha^2\partial\beta}\right]$	$c^2\nabla^2\nabla^2 + 1$	$-\frac{1-\nu^3}{E\delta}R^2Z$

(8.12)

In comparing the equations of table 3 with equations (8.7) (given in table 2), it is noted that in equations (8.12) there are absent, in the first place, certain terms arising from the moment, namely, the terms with $c^2 = \delta^2/(12R^2)$; in the second place, the differential operators (of the second of the third equation and third of the second) are asymmetric. The absence of symmetry in equations (8.12) is in contradiction to the fundamental theorems of the energostatic elastic body. For this reason, the existing theory of shells starts from a number of classical problems of the mechanics of elastic bodies.

The previously mentioned defects of equations (8.12) may lead to a fundamental error in the problem of the vibration of shells. Given any three independent forms of vibration with the corresponding displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ and applying the method of Galerkin to equations (8.12), there is obtained for the frequency of the vibrations a cubical equation, which being represented in the form of a determinant of the third order (corresponding to the mechanical significance of the problem) has an asymmetric structure. Due to this asymmetry, two of the vibration frequencies for arbitrarily chosen forms of $\bar{u}$, $\bar{v}$, and $\bar{w}$ may receive imaginary values, a result that is likewise in contradiction to the theory of small vibrations of elastic bodies.

The absence of symmetry in the equations of the moment theory of cylindrical shells was noted in previously published papers on the theory of shells and thin rods (reference 10). In these papers are given equations of the strength, the stability, and the vibrations of shells of composite systems and rods possessing collateral auxiliary differential operators of the required functions of symmetric structure. The recent works on shells (reference 7), which improve to a greater or less degree the moment theory, suffer from the defects pointed out here.

9. Spherical Shell. Generalization of Equation of Sophie-Germain - Lagrange. - In this case,

$$\left.\begin{aligned} k_1 = k_2 = k = 1/R &= \text{constant} \\ H &= k \\ L &= 0 \end{aligned}\right\} \tag{9.1}$$

From equations (5.8),

$$\left.\begin{aligned}\Delta_0 &= \theta + 2kw + w^* \\ \Delta_1 &= -\nabla^2 w - 2k^2 w + 2kw^* \\ \Delta_2 &= -k\Delta_1 \\ \chi_1 &= \chi_2 = 0\end{aligned}\right\} \tag{9.2}$$

The system (5.4) leads to an equivalent system of the form

$$\left.\begin{aligned}&(\lambda+2\mu)\nabla^2\theta + 2\mu k^2\theta + 2(\lambda+\mu)k\nabla^2 w - (\lambda+2\mu)\frac{\delta^2}{12}k\nabla^2\nabla^2 w + \lambda\nabla^2 w^* = \\ &\qquad -\frac{1}{\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right] \\ &2(\lambda+\mu)k\theta - (\lambda+2\mu)\frac{\delta^2}{12}k\nabla^2\theta + (\lambda+2\mu)\frac{\delta^2}{12}\nabla^2\nabla w^2 + 2\mu\frac{\delta^2}{12}k^2\nabla^2 w + \\ &\qquad 4(\lambda+\mu)k^2 w - (3\lambda+2\mu)\frac{\delta^2}{12}k\nabla^2 w^* + 2\lambda k w^* = \frac{1}{\delta}Z \\ &\lambda\theta - (3\lambda+2\mu)\frac{\delta^2}{12}k\nabla^2 w + 2\lambda k w - \mu\frac{\delta^2}{12}\nabla^2 w^* + (\lambda+2\mu)w^* = \frac{1}{\delta}Z^* \\ &\nabla^2\chi + k^2\chi = -\frac{1}{2\mu\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BY) - \frac{\partial}{\partial\beta}(AX)\right]\end{aligned}\right\} \tag{9.3}$$

where θ is the volume dilation of the shell for the tangential deformation and 2χ is the normal rotation:

$$\left.\begin{aligned}\theta &= \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu) + \frac{\partial}{\partial\beta}(Av)\right] \\ \chi &= \frac{1}{2AB}\left[\frac{\partial}{\partial\alpha}(Bv) - \frac{\partial}{\partial\beta}(Au)\right]\end{aligned}\right\} \tag{9.4}$$

The symbol ∇^2 is the differential operator of the second order (operator of Beltrami for the sphere):

$$\nabla^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right] \tag{9.5}$$

In deriving equations (9.3), $1 + k^2\delta^2/12 \approx 1$ is assumed because of the smallness of the term $k^2\delta^2/12$ as compared with one.

The first three equations of (9.3) form a complete system having a symmetric structure with respect to the functions θ, $\nabla^2 w$, and $\nabla^2 w^*$. The fourth equation, independently of the first three, determines the normal elongation.

In the case of a closed spherical shell under the action of normal rotation on the inner and outer surfaces for constant (independent of α,β) intensities of these pressures, the differential terms drop out, $X = Y = 0$ and then

$$\left.\begin{aligned} \theta = \chi &= 0 \\ 4(\lambda+\mu)k^2 w + 2\lambda k w^* &= \frac{Z}{\delta} \\ 2\lambda k w + (\lambda+2\mu)w^* &= \frac{Z^*}{\delta} \end{aligned}\right\} \tag{9.6}$$

where by virtue of equations (5.5) and (5.7)

$$\left.\begin{aligned} Z &= \left|(1+k\gamma)^2\,\sigma_\gamma\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \\ Z^* &= \left|(1+k\gamma)^2\,\gamma\sigma_\gamma\right|_{-\frac{1}{2}\delta}^{+\frac{1}{2}\delta} \end{aligned}\right\} \tag{9.7}$$

If w^* is set equal to zero and correspondingly the third of equations (9.3) is neglected, then for $\lambda = E\nu/(1-\nu^2)$, $\mu = E/2(1+\nu)$

$$\left.\begin{aligned}
&\nabla^2\theta + (1-\nu)k^2\theta - \frac{\delta^2}{12} k\nabla^2\nabla^2 w + (1+\nu)k\nabla^2 w = -\frac{1-\nu^2}{E\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right]\\
&-\frac{\delta^2}{12} k\nabla^2\theta + (1+\nu)k\theta + \frac{\delta^2}{12}\nabla^2\nabla^2 w + (1-\nu)\frac{\delta^2}{12}k^2\nabla^2 w + 2(1+\nu)k^2 w = \frac{1-\nu^2}{E\delta} Z\\
&\nabla^2\chi + k^2\chi = -\frac{1+\nu}{E\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BY) - \frac{\partial}{\partial\beta}(AX)\right]
\end{aligned}\right\} \tag{9.8}$$

For $k = 0$, the first two of these equations break down into the equation of Lamé

$$\nabla^2\theta = -\frac{1-\nu^2}{E\delta}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BY) + \frac{\partial}{\partial\beta}(AX)\right] \tag{9.9}$$

which refers to the problem of the two-dimensional stress state of a plate of thickness δ, and the equation of Sophie-Germain-Lagrange

$$\nabla^2\nabla^2 w = \frac{1}{D} Z \qquad \left(D = \frac{E\delta^3}{12(1-\nu^2)}\right) \tag{9.10}$$

which refers to the problem of the bending of a plate.

By eliminating the function θ from the first two equations of (9.8), there is obtained

$$\left.\begin{aligned}
&\frac{\delta^2}{12(1-\nu^2)}\left(\nabla^2\nabla^2\nabla^2 w + 4k^2\nabla^2\nabla^2 w\right) + k^2\left(\nabla^2 w + 2k^2 w\right)\\
&= \frac{1}{E\delta}\Bigg\{(1+\nu)\frac{k}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right] -\\
&(1-\nu)k^2 Z + \nabla^2\left(Z - \frac{\delta^2}{12}\frac{k}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right]\right)\Bigg\}
\end{aligned}\right\} \tag{9.11}$$

Equation (9.11) is the fundamental equation of the spherical shell with inextensible normal element and constitutes a natural generalization of the equation of Sophie Germain-Lagrange for the bending of a plate.

Having determined the deflection w and the normal expansion from the third of equations (9.8), the tangential displacements u and v can be determined by the equations

$$\left.\begin{aligned} u &= -\frac{1}{(1-\nu)k^2}\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\theta + 2kw - \frac{\delta^2}{12}k\nabla^2 w\right) + \frac{1}{k^2}\frac{1}{B}\frac{\partial X}{\partial\beta} + \frac{1}{k}\frac{1}{A}\frac{\partial w}{\partial\alpha} - \frac{1+\nu}{E\delta k^2}X \\ v &= -\frac{1}{(1-\nu)k^2}\frac{1}{B}\frac{\partial}{\partial\beta}\left(\theta + 2kw - \frac{\delta^2}{12}k\nabla^2 w\right) - \frac{1}{k^2}\frac{1}{A}\frac{\partial X}{\partial\alpha} + \frac{1}{k}\frac{1}{B}\frac{\partial w}{\partial\beta} - \frac{1+\nu}{E\delta k^2}Y \end{aligned}\right\} \tag{9.12}$$

where

$$\left.\begin{aligned} \theta = &-\frac{\delta^2}{12(1+\nu)k}\nabla^2\nabla^2 w - \frac{\delta^2 k}{6(1+\nu)}\nabla^2 w - 2kw - \\ &\frac{(1-\nu)\delta}{12E}\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BX) + \frac{\partial}{\partial\beta}(AY)\right] + \frac{1-\nu}{E\delta k}Z \end{aligned}\right\} \tag{9.13}$$

The theory of the spherical shell for thick shells (equations (9.3)) as well as for moderately thick shells (equations (9.8) and (9.11)) have been presented. The fundamental functions chosen θ, w, X, and w^* are invariant relative to the direction of the coordinate lines α,β passing through a given point on the sphere.

It follows that the equations given are valid for any system of coordinates on the spherical surface. The choice of coordinates determines only the differential operator ∇^2. If for the coordinates α,β the geographical coordinates were taken, taking α as the latitude and β as the longitude, then for $k = 1/R$, $A = R$ and $B = R \sin \alpha$ so that

$$\nabla^2 = \frac{1}{R^2}\left(\frac{\partial^2}{\partial\alpha^2} + \cot\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\beta^2}\right) \tag{9.14}$$

For an arbitrary load (nonsymmetrical problem), equations (9.3) and in the particular case (for $w^* = 0$) equations (9.8) or (9.11) are integrated by the method of separation of variables with respect to the variable β in trigonometric functions and with respect to the variable α in Legendre functions.

10. General Equations for Stability of Shells - Special Cases. - It is assumed that the shell has a given system of stresses characterized only by the tangential (normal and shearing) forces $T_1{}^0$, $T_2{}^0$, and S^0 and in equilibrium with the external forces. It shall be considered that the external forces are given with an accuracy up to one parameter, for example, the intensity of any of the components of the external load. By assigning different values to this parameter, different stress states are obtained.

In a particular case, the internal forces $T_1{}^0$, $T_2{}^0$, and S^0 may be proportional to the intensity of the external load. For a certain value of the load parameter, the equilibrium of the shell becomes unstable.

From the stress state $T_1{}^0$, $T_2{}^0$, and S^0, the shell passes to another state $T_1{}^0 + T_1$, $T_2{}^0 + T_2$, $S^0 + S$, G_1, G_2, H, N_1, and N_2 where T_1, T_2, . . . , N_2 are internal forces arising on the loss of stability. It shall be assumed that the forces T_1, T_2, . . . , N_2 and the corresponding deformations are infinitely small magnitudes. Because the change in the deformed state of the shell, associated with loss in stability, is characterized by a change in form of the middle surface, it is necessary, in order to obtain the equations of stability, to take into account the variations of the magnitudes referring only to the second-quadratic form of the surface.

The stability equation is obtained from the equations (6.2) and (6.3) given for shells of medium thickness or from equations (7.4) for thin shells. It is necessary in the first place to refer all static and kinematic magnitudes entering these equations to variations of the stress and the deformed state of the shell that arise on loss of stability and in the second place to consider the components X, Y, and Z as those surface forces that are obtained when an element of the shell $AB\,d\alpha\,d\beta$ with the contour forces $T_1{}^0 + T_1$, $T_2{}^0 + T_2$, $S^0 + S$, . . . is carried into the new deformed state determined by the displacements u, v, and w.

With the passage of the shell into the deformed state, the normal to the middle surface will have a new direction determined by the angles of rotation

$$\left.\begin{aligned} q_1 &= k_1 u - \frac{1}{A}\frac{\partial w}{\partial \alpha} \\ q_2 &= -\left(k_2 v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right) \end{aligned}\right\} \tag{10.1}$$

relative to the tangents to the lines α = constant and β = constant of the initial state. For the components X, Y, and Z of the vector of the reduced surface force on the axes of coordinates α, β, and γ of the movable trihedron of the middle surface, small magnitudes are readily obtained with an accuracy up to second order

$$\left.\begin{aligned}
X &= -k_1\left[\left(k_1u - \frac{1}{A}\frac{\partial w}{\partial\alpha}\right)T_1{}^0 + \left(k_2v - \frac{1}{B}\frac{\partial w}{\partial\beta}\right)S^0\right] \\
Y &= -k_2\left[\left(k_2v - \frac{1}{B}\frac{\partial w}{\partial\beta}\right)T_2{}^0 + \left(k_1u - \frac{1}{A}\frac{\partial w}{\partial\alpha}\right)S^0\right] \\
Z &= -\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\left(B\left[\left(k_1u - \frac{1}{A}\frac{\partial w}{\partial\alpha}\right)T_1{}^0 + \left(k_2v - \frac{1}{B}\frac{\partial w}{\partial\beta}\right)S^0\right]\right) - \right. \\
&\qquad \left.\frac{\partial}{\partial\beta}\left(A\left[\left(k_2v - \frac{1}{B}\frac{\partial w}{\partial\alpha}\right)T_2{}^0 + \left(k_1u - \frac{1}{A}\frac{\partial w}{\partial\alpha}\right)S^0\right]\right)\right\}
\end{aligned}\right\} \quad (10.2)$$

By substituting the values of X, Y, and Z thus obtained in equations (6.2) and neglecting in these equations m_α and m_β, the general equation of stability of the shells is obtained. In the case of a thin shell, X, Y, and Z must be substituted in equations (7.4).

In either case, there is obtained with the accuracy of the load parameter a complete system of homogeneous differential equations in the required functions that determine the deformations of the shell associated with loss in stability. To this system are added the boundary conditions (homogeneous).

The critical stress state $T_1{}^0$, $T_2{}^0$, and S^0 determined by the parameter of the external load entering linearly in equations (10.2) is thus determined by solving the homogeneous boundary problem described here.

Inasmuch as of the three variables u, v, and w the normal displacement w has the principal effect on the change in shape of the shell, in equations (10.2) the terms with the tangential displacements u and v may be neglected. Then

$$\left.\begin{aligned}
X &= k_1\left(T_1{}^0 \frac{1}{A}\frac{\partial w}{\partial \alpha} + S^0 \frac{1}{B}\frac{\partial w}{\partial \beta}\right) \\
Y &= k_2\left(T_2{}^0 \frac{1}{B}\frac{\partial w}{\partial \beta} + S^0 \frac{1}{A}\frac{\partial w}{\partial \alpha}\right) \\
Z &= \frac{1}{AB}\left\{\frac{\partial}{\partial \alpha}\left(T_1{}^0 \frac{B}{A}\frac{\partial w}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(T_2{}^0 \frac{B}{A}\frac{\partial w}{\partial \beta}\right) + \frac{\partial}{\partial \alpha}\left(S^0 \frac{\partial w}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(S^0 \frac{\partial w}{\partial \alpha}\right)\right\}
\end{aligned}\right\} \tag{10.3}$$

In the case of a thin shell, the components X and Y, being proportional to the curvatures k_1 and k_2, may be taken equal to zero. On the basis of equations (7.8)

$$\left.\begin{aligned}
&\frac{1}{E\delta}\nabla_e{}^2\nabla_e{}^2\Phi - (H\nabla_e{}^2 - L\nabla_h{}^2)w = 0 \\
&-(H\nabla_e{}^2 - L\nabla_h{}^2)w - \frac{E\delta^3}{12(1-\nu^2)}\nabla_e{}^2\nabla_e{}^2 w + \frac{1}{AB}\left[\frac{\partial}{\partial \alpha}\left(T_1{}^0 \frac{B}{A}\frac{\partial w}{\partial \alpha}\right) + \right. \\
&\left.\frac{\partial}{\partial \beta}\left(T_2{}^0 \frac{A}{B}\frac{\partial w}{\partial \beta}\right) + \frac{\partial}{\partial \alpha}\left(S^0 \frac{\partial w}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(S^0 \frac{\partial w}{\partial \alpha}\right)\right] = 0
\end{aligned}\right\} \tag{10.4}$$

These equations constitute the general equations of stability of thin shells in the two functions Φ and w and permit determining the critical stresses for very general assumptions with regard to the given stress state.

The general theory of the stability of shells has been presented exactly as given by equations (10.2), (6.2), and (6.3) and approximately as given by equations (10.3), (7.4) or (10.4).

This theory represents a considerable generalization of a number of problems on stability of elastic systems, starting with the simplest problem of longitudinal bending and ending with the stability of shells of arbitrary shape for arbitrarily given initial stress state $T_1{}^0 = T_1{}^0(\alpha,\beta)$, $T_2{}^0 = T_2{}^0(\alpha,\beta)$, and $S^0 = S^0(\alpha,\beta)$, the critical value of which is determined.

Thus, for example, in equations (10.4) setting $k_1 = k_2 = 0$, from the second equation (the operator

$H\nabla_e^2 - L\nabla_h^2 = \frac{k_1+k_2}{2} \nabla_e^2 - \frac{k_1-k_2}{2} \nabla_h^2$ becomes zero), the equation of stability of a plate in arbitrary orthogonal curvilinear coordinates α and β are obtained. For

$$\left.\begin{aligned} A = B = 1 \\ \frac{\partial T_1{}^0}{\partial \alpha} + \frac{\partial S^0}{\partial \beta} = 0 \\ \frac{\partial T_2{}^0}{\partial \beta} + \frac{\partial S^0}{\partial \alpha} = 0 \end{aligned}\right\}$$

there is obtained

$$\frac{E\delta^3}{12(1-\nu^2)} \nabla^2\nabla^2 w - T_1{}^0 \frac{\partial^2 w}{\partial \alpha^2} - T_2{}^0 \frac{\partial^2 w}{\partial \beta^2} - 2S^0 \frac{\partial^2 w}{\partial \alpha \partial \beta} = 0 \qquad (10.5)$$

Equation (10.5) is the well-known equation, in rectilinear coordinates, of the stability of a plate loaded by forces on the boundary.

For $k_1 = k_2 = k = 1/R =$ constant, $S^0 = 0$, and $T_1{}^0 = T_2{}^0 = -pR/2 =$ constant, the following equation is obtained:

$$\frac{R^2\delta^2}{12(1-\nu^2)} \nabla_e{}^2 \nabla_e{}^2 \nabla_e{}^2 w + \nabla_e{}^2 w + \frac{pR^3}{2E\delta} \nabla_e{}^2 w = 0 \qquad (10.6)$$

which refers to the stability of a spherical shell of radius R, under an internal pressure $p =$ constant, the parameter of this equation being p.

A cylindrical shell will be considered, starting at first from the more accurate equations (10.2) and (8.7). If α and β are the absolute coordinates, $A = B = 1$. Equations (10.2) for $k_1 = 0$ and $k_2 =$ constant become

$$\left.\begin{aligned}
X &= 0\\
Y &= -k_2\left[T_2{}^0\left(k_2 v - \frac{\partial w}{\partial \beta}\right) - S^0\frac{\partial w}{\partial \alpha}\right]\\
Z &= \frac{\partial}{\partial \alpha}\left[T_1{}^0\frac{\partial w}{\partial \alpha} - S^0\left(k_2 v - \frac{\partial w}{\partial \beta}\right)\right] + \frac{\partial}{\partial \beta}\left[S^0\frac{\partial w}{\partial \alpha} - T_2{}^0\left(k_2 v - \frac{\partial w}{\partial \beta}\right)\right]
\end{aligned}\right\} \tag{10.7}$$

If $\dfrac{\partial T_1{}^0}{\partial \alpha} + \dfrac{\partial S^0}{\partial \beta} = 0$, $\dfrac{\partial T_2{}^0}{\partial \beta} + \dfrac{\partial S^0}{\partial \alpha} = 0$, and $A = B = R$ (hence α and β are relative coordinates), equations (10.7) become the following:

$$\left.\begin{aligned}
X &= 0\\
Y &= -\frac{1}{R^2}\left[T_2{}^0\left(v - \frac{\partial w}{\partial \beta}\right) - S^0\frac{\partial w}{\partial \alpha}\right]\\
Z &= \frac{1}{R^2}\left[T_1{}^0\frac{\partial^2 w}{\partial \alpha^2} - T_2{}^0\frac{\partial}{\partial \beta}\left(v - \frac{\partial w}{\partial \beta}\right) + S^0\left(2\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{\partial v}{\partial \alpha}\right)\right]
\end{aligned}\right\} \tag{10.8}$$

On the basis of equations (10.8), table 2 (equations (8.7)) assumes the form given by table 4. Equations (10.9) (table 4) in the secondary operators possess also in this case a symmetrical structure, a fact that as already noted is in agreement with the theorem of reciprocity and therefore the critical forces will always be real.[1]

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[1]The equations that are used by Timoshenko (reference 3) and other authors (references 7 and 8) are assymetric with respect to the secondary terms and consequently do not correspond to the fundamental theorems of the theory of elasticity.

TABLE 4

$u(\alpha,\beta)$	$v(\alpha,\beta)$	$w(\alpha,\beta)$
$\frac{\partial^2}{\partial\alpha^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2}$	$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\nu\frac{\partial}{\partial\alpha} - c^2\left(\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right)$
$\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta}$	$\frac{\partial^2}{\partial\beta^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2} - (1-\nu^2)\frac{T_2{}^0}{E\delta}$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}c^2\frac{\partial^3}{\partial\alpha^2\partial\beta} + \frac{1-\nu^2}{E\delta}\left(T_2{}^0\frac{\partial}{\partial\beta} + S^0\frac{\partial}{\partial\alpha}\right)$
$\nu\frac{\partial}{\partial\alpha} - c^2\left(\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right)$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}c^2\frac{\partial^3}{\partial\alpha^2\partial\beta} + \frac{1-\nu^2}{E\delta}\left(T_2{}^0\frac{\partial}{\partial\beta} + S^0\frac{\partial}{\partial\alpha}\right)$	$c^2\left(\Delta^2\Delta^2 + 2\frac{\partial^2}{\partial\beta^2} + 1\right) - \frac{1-\nu^2}{E\delta}\left(T_1{}^0\frac{\partial^2}{\partial\alpha^2} + T_2{}^0\frac{\partial^2}{\partial\beta^2} + 2S\frac{\partial^2}{\partial\alpha\partial\beta}\right)$

(10.9)

These equations are the general equations of stability of a circular shell, obtained in strict correspondence with the fundamental hypothesis of Kirchhoff - Love and make it possible to consider a number of problems of practical interest on the determination of the critical loads of the shell.

Because the components X, Y, and Z determined in the general case of the stress state by equations (10.8) are obtained with account taken of the exact values of the angles of rotation q_1 and q_2 of the normal of the shell, equations (10.9) are applicable also to shells of medium thickness.

In the case of a shell for which $\delta/R \leqslant 1/3^0$, the tangential contour force Y represents, according to equations (10.7), a magnitude that is small in comparison with the normal force Z. The tangential contour displacement v on deformation of the shell accompanied by the change in shape of the cross section is a magnitude that is likewise small compared with the normal displacement w. By assuming for a thin shell the magnitude Y to be equal to zero and neglecting in the last of equations (10.8) the tangential displacement v,

$$Z = \frac{1}{R^2}\left[\frac{\partial}{\partial\alpha}\left(T_1{}^0\frac{\partial w}{\partial\alpha} + S^0\frac{\partial w}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\left(T_2{}^0\frac{\partial w}{\partial\beta} + S^0\frac{\partial w}{\partial\alpha}\right)\right] \quad (10.10)$$

The general stability equation of a shell for given assumptions as to the force Y may be obtained from equation (8.10) by substituting in this equation the value of Z determined by equation (10.10). This equation has the form[2]

$$\left.\begin{aligned}&c^2(\nabla^4+2\nabla^2+1)\nabla^4\Phi - 2c^2(1-\nu)\left(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\nabla^2\Phi + (1-\nu^2)\frac{\partial^4\Phi}{\partial\alpha^4} - \\ &\frac{1-\nu^2}{E\delta}\left\{\frac{\partial}{\partial\alpha}\left(T_1{}^0\frac{\partial}{\partial\alpha}\nabla^4\Phi + S^0\frac{\partial}{\partial\beta}\nabla^4\Phi\right) + \frac{\partial}{\partial\beta}\left(T_2{}^0\frac{\partial}{\partial\beta}\nabla^4\Phi + S^0\frac{\partial}{\partial\alpha}\nabla^4\Phi\right)\right\} = 0\end{aligned}\right\} \quad (10.11)$$

[2]If in equations (10.11) the second and third terms of the first component and the complete second component are neglected, the approximate equation for the stability of a thin cylindrical shell shall be obtained.

Equation (10.11) for these assumptions is equivalent to the system of equations (10.9) in the three functions. The displacements u, v, and w are determined in terms of the fundamental function Φ by equation (8.9). It should be noted that $T_2^0 = S^0 = 0$, that is, in the case of the stress state characterized only by longitudinal normal forces T_1^0 (central compression, for example, pure bending, eccentric action of longitudinal compressive or tensile forces, and so forth), these assumptions drop out. The equations given here are the general equations of the stability of a cylindrical shell from which the critical stress can be determined for very different assumptions both as regards the given external forces and as regards the boundary conditions. Thus, for example, the equations of stability can be obtained for the following cases:

1. Central compression of a shell by a force P

$$\left.\begin{aligned} T_2^0 &= S^0 = 0 \\ T_1^0 &= -\frac{P}{2\pi R} \end{aligned}\right\}$$

2. Pure torsion of a closed shell

$$\left.\begin{aligned} T_1^0 &= T_2^0 = 0 \\ S^0 &= \frac{M}{2\pi R^2} \end{aligned}\right\}$$

3. Shells under the action of an external normal pressure and immovably clamped at the longitudinal edge against displacements u and w

$$\left.\begin{aligned} T_1^0 &= S^0 = 0 \\ T_2^0 &= -qR \end{aligned}\right\}$$

4. Shells under the simultaneous action of a longitudinal compressive (or tensile) force P and twisting moment M

$$\left.\begin{aligned} T_2^0 &= 0 \\ T_1^0 &= \mp\frac{P}{2\pi R} \\ S^0 &= \frac{M}{2\pi R^2} \end{aligned}\right\}$$

In this case for the load parameter, there may be taken the magnitude P for a given value of the magnitude M or M for a given value of P, or the ratio of these values as a function of the conditions of the problem

5. Shells under the action of only a single bending moment (pure bending) or of a moment and a longitudinal force (bending with tension or compression)

6. Shells under the action of a transverse load producing at the sections α = constant longitudinal, normal, and shearing forces T_1^0 and S^0, determined by the usual elementary theory of the bending of beams, and so forth

In all of these cases except cases 5 and 6 the differential equations of stability have constant coefficients.

The critical stresses are determined by solving the homogeneous boundary problem by equation (10.11) or in the case of a more accurate solution by the system of equations (10.9) and the homogeneous boundary conditions. If the shell of length l on each of the curvilinear edges $\alpha = 0$ and $\alpha = l/R$ is hinge-supported on a diaphragm that is rigid in its plane and flexible in the transverse plane, the function Φ corresponding to these boundary conditions may be approximated for the closed shell in the form of a double trigonometric series:

$$\Phi = \sum \sum A_{mn} \sin \frac{m\pi R\alpha}{l} \cos n\beta \qquad (m,n = 1, 2, 3, \ldots)$$

and for shells of open profile in the form of a trigonometric series in only one variable α:

$$\Phi = \sum \Psi_m(\beta) \sin \frac{m\pi R\alpha}{l} \qquad (m = 1, 2, 3, \ldots)$$

where the function $\Psi_m(\beta)$ is determined by ordinary differential equations (homogeneous with one parameter) and the boundary condition (likewise homogeneous), which must be given on the straight edges of the shell.

REMARKS

The theory given is of general character and permits solving a number of practically important problems on the strength, the stability, and the vibrations of shells. Thus for example:

1. Computation of shells by the method of the theory of complex variables. - In reference 12, it is shown that for shells characterized by middle surfaces of the second order with positive Gaussian curvature (spherical, elliptical, and parabolical) the equations of the momentless theory characterized by the mixed differential operator $H\nabla_e^2 - L\nabla_h^2$ leads, through transformation of the independent variables, to the Cauchy-Reiemann equations. These investigations show that the more accurate equations (7.8) relative to the moment theory of thin shells will be of the elliptic type for middle surfaces of the second order. These equations for such surfaces also lead to the equations of Cauchy-Riemann. It then follows that the computation of such shells by the moment theory may be effected by the methods of the theory of functions of a complex variable by developing and generalizing the known methods of Muskhelishvili (reference 13) on the two-dimensional problem of the theory of elasticity. In particular, it is of interest to determine the stresses and the deformations of shells of spherical, elliptic, and parabolic types due to the action of a concentrated force applied at any point of the middle surface.

2. Circular cylindrical shell under the action of a concentrated force. - The solution of this problem may be obtained by the integration of equation (8.10) or for the thin shell of equations (7.8) by the method of separation of variables. (In this case $H\nabla_e^2 - L\nabla_h^2 = k_2 \frac{\partial^2}{\partial\alpha^2}$.) The functions required may be approximated by trigonometric series in one of the variables α or β, as in the method of Failon-Ribier for the two-dimensional stress state of a rectangular plate and in the method of Morris-Levy for the case of the bending of such a plate. The Green Function may be represented by a Fourier Integral.

3. Tension in a closed circular shell having somewhere on the surface an opening of given shape.

4. Torsion of a circular shell weakened by an opening. - Both of these problems may be solved also with the aid of trigonometric series.

5. Stability of an open circular shell in the case: (a) central compression, (b) pure bending, (c) compression with bending, and (d) bending by given transverse forces. - This problem is solved by applying to the stability equations given ordinary trigonometric series in the variable along the generator.

6. Stability of a closed circular shell in torsion. - The required functions in this case can be given in the form of trigonometric series in the variable β (in the direction of the transverse arc).

7. Stability of a spherical shell under the action of an external hydrostatic pressure. - The differential equation corresponding to this problem can be integrated by the method of separation of the variables by applying trigonometric functions and functions of Legendre.

Translated by S. Reiss
National Advisory Committee
for Aeronautics.

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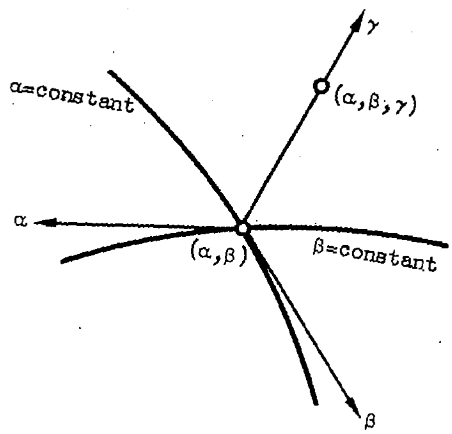


Figure 1

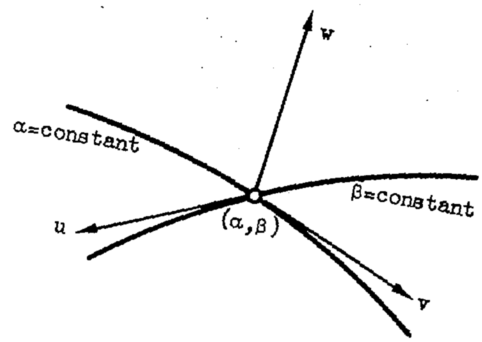


Figure 2

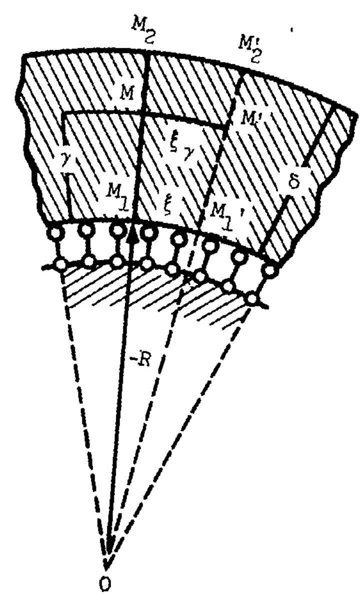
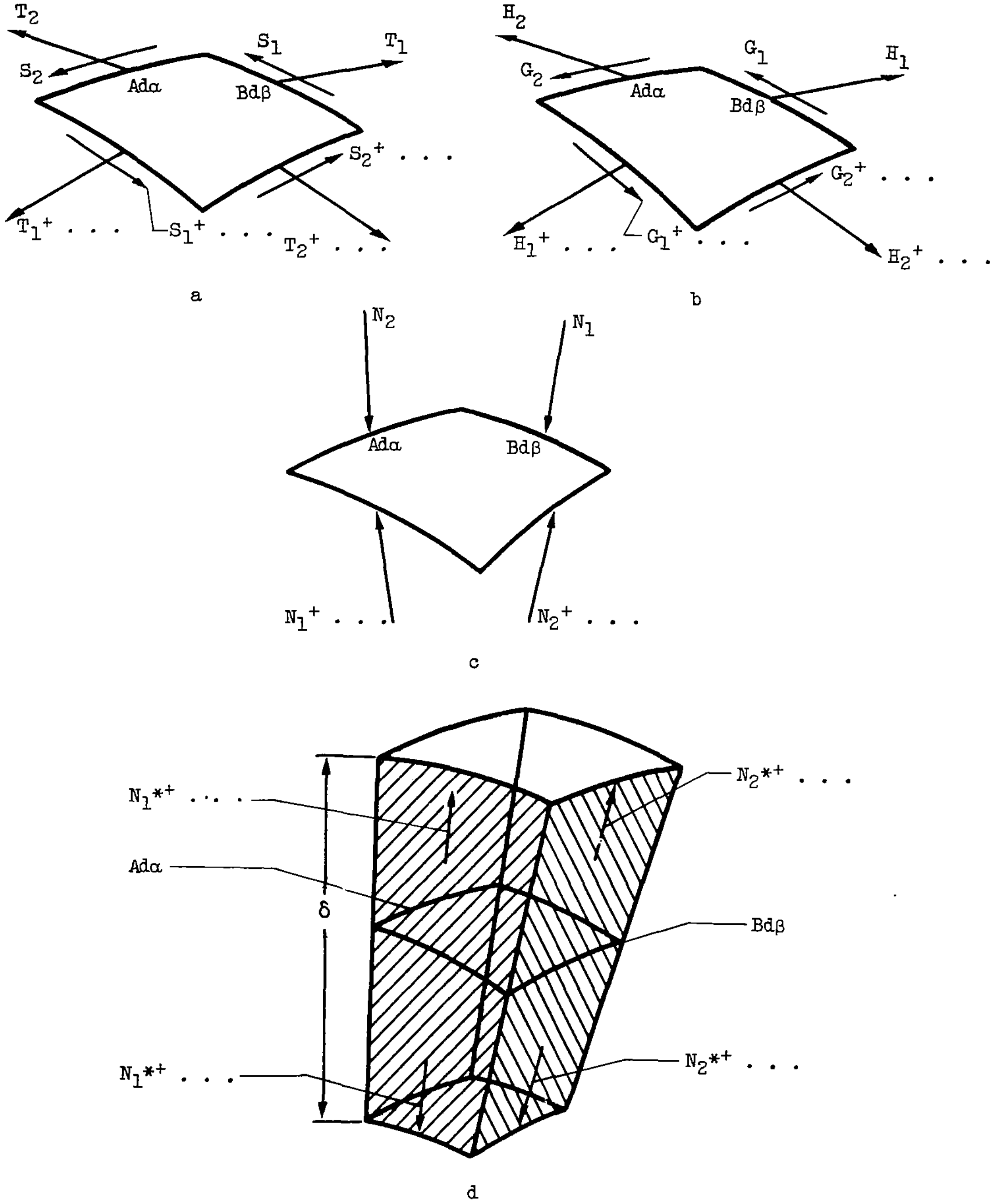


Figure 3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N1
Adα
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Figure 4